

Question 1

Question n

Suppose that there are 50 million people in the labor force in Mexico and 150 million in the U.S. All workers prefer to work in their own country, but the extent of this preference varies across people. Assume that Mexican and U.S. workers are perfect substitutes (that is, they are equally productive when working in the same country), and assume that the same product is produced in both countries, and that the product price is 1.

The technology in each country is described by a Cobb-Douglas production function with constant returns:

$$Q_i = A_i K_i^\alpha L_i^{1-\alpha}$$

for $i \in \{1, 2\}$, with $\alpha \in (0, 1)$.

The total supply of capital in the two countries is a fixed amount K^0 , and capital can be moved from one country to the other at no cost. The owners of the capital act so as to maximize income – the capital is rented to the highest bidder.

All markets are competitive, except that there may be restrictions on migration of labor from one country to the other.

- A Suppose that immigration is not allowed, and it is observed that the equilibrium wage is \$30 per hour in the U.S., and \$10 per hour in Mexico. Does this imply that U.S. firms are more productive (i.e. that $A_1 > A_2$)? If $\alpha = \frac{1}{3}$, and $A_2 = 1$, do you have enough information to determine A_1 ?
- B Now suppose that workers can freely migrate from one country to the other. The home country preference depends on the relative wage, $\omega = \frac{w_1}{w_2}$, and preferences are uniformly distributed. If $\omega = 5$, all Mexican workers would prefer to work in the U.S.; and if $\omega = 2$, 40% of Mexican workers would prefer to work in the U.S., and so on.
- How many people will migrate, in the new equilibrium?
 - What happens to wages in each country? Explain why.
 - Does the relative wage rise or fall?
 - What happens to output in each country? What happens to total output (the sum of the outputs in the two countries)?
 - Discuss the welfare implications of your results.

Answer

If capital is freely mobile across two countries but labor is not, then in equilibrium the marginal product of capital is the same in the two countries, and since marginal and average products are proportional for the Cobb-Douglas technology, this implies

$$\frac{Q_1}{K_1} = \frac{Q_2}{K_2}$$

and

$$A_1 k_1^{\alpha-1} = A_2 k_2^{\alpha-1}$$

and

$$\kappa = a^\theta$$

where $k_i = \frac{K_i}{L_i}$ is the capital-labor ratio, and $\kappa = \frac{k_1}{k_2}$, and $a = \frac{A_1}{A_2}$, and $\theta = \frac{1}{1-\alpha}$.

Wages are equal to the marginal product of labor

$$w_i = (1 - \alpha) A_i k_i^\alpha,$$

for $i \in \{1, 2\}$. So the wage ratio is

$$\begin{aligned} \omega &= a\kappa^\alpha \\ &= a^\theta \end{aligned}$$

which implies $\omega = \kappa$.

If $\omega = 3$ and $\alpha = \frac{1}{3}$, then $\theta = \frac{3}{2}$, and $a = 3^{\frac{2}{3}} \approx 2.08$, so if $A_2 = 1$ then $A_1 = 3^{\frac{2}{3}}$ (and U.S. firms are indeed more productive).

The relative wage does not depend on the number of workers in each country (at an interior solution). So in the equilibrium with migration, 30 million Mexican workers move to the U.S.

Since κ is determined by the technology, the equilibrium allocation for any given values of L_1 and L_2 is obtained by solving

$$\begin{aligned} K_1 + K_2 &= K^0 \\ \frac{K_1}{L_1} &= \kappa \frac{K_2}{L_2} \end{aligned}$$

The solution is

$$K_1 = \frac{1}{1 + \frac{L_2}{\kappa L_1}} K^0$$
$$K_2 = \frac{1}{1 + \frac{\kappa L_1}{L_2}} K^0$$

The capital-labor ratio in the U.S. is then given by

$$k_1 = \frac{\kappa}{\kappa L_1 + L_2} K^0$$
$$= \frac{\kappa}{(\kappa - 1) L_1 + L^0} K^0$$

where L^0 is the total number of workers (including both countries). Since $\kappa > 1$, immigration reduces the capital-labor ratio in the U.S., so wages fall in the U.S. And wages also fall in Mexico, because the relative wage does not change.

Total output increases when workers move to the more productive location. Maximal output would be obtained by moving all workers to the U.S., but this is not efficient.

The equilibrium with no migration is not efficient; the equilibrium with free migration is efficient (by the first welfare theorem). Migration makes all non-migrants worse off (and this includes all U.S. workers); some (but not all) migrants are better off in the new equilibrium.

Question 2

Consider a pure exchange economy with two goods, $h = 1, 2$, and two consumers, $i = 1, 2$, with utility functions u_1 and u_2 respectively, and aggregate endowment, $w = (w_1, w_2) \gg 0$. For each of the following cases, determine which of the Pareto-efficient allocations can be decentralized as competitive equilibrium with lump sum transfers of wealth. Briefly describe the equilibrium prices and transfers for each Pareto-efficient allocation. [Hint: An equilibrium with transfers involves finding (non-negative) prices (p_1, p_2) and transfers (t_1, t_2) satisfying $t_1 + t_2 = 0$, such that consumers maximize utility at the chosen allocation given budget constraints $p_1 x_{i1} + p_2 x_{i2} \leq p_1 w_{i1} + p_2 w_{i2} + t_i$, $i = 1, 2$.]

- (a) (4 points) $u_1(x_{11}, x_{12}) = \alpha \ln x_{11} + (1 - \alpha) \ln x_{12}$ and $u_2(x_{21}, x_{22}) = \beta \ln x_{21} + (1 - \beta) \ln x_{22}$, where $0 < \alpha < \beta < 1$.

ANSWER: In this case, for any Pareto-efficient allocation $(x_{11}^*, x_{12}^*), (x_{21}^*, x_{22}^*)$, we know that at least one of the two agents is consuming a positive amount of the two goods. Otherwise, if each agent only consumes one good their marginal utilities with respect to the good they don't consume are infinite. Hence, both could be strictly better off by trading some of the good they get for a little amount of the other good. But then, by a similar argument, an allocation in which one consumer gets positive amounts of both while the other gets a positive amount of only one of the goods cannot be Pareto optimal. This is because, in such a case, she would be willing to sacrifice most of her consumption of the good she is getting for a tiny amount of the good she is not getting, making both consumers better off—the consumer getting both goods has finite marginal utilities at the proposed allocation and therefore she will accept such a trade. Thus, either we have an interior allocation or one of the consumers is getting the aggregate endowment of both goods. Assume consumer i gets positive amounts of both goods, then her FOC for utility maximization implies that

$$\frac{p_1}{p_2} = \frac{\partial u_i(x_{11}^*, x_{12}^*) / \partial x_{11}}{\partial u_i(x_{11}^*, x_{12}^*) / \partial x_{12}} = \frac{\alpha}{1 - \alpha} \frac{x_{12}^*}{x_{11}^*}.$$

If the allocation is interior then we have that

$$\frac{p_1}{p_2} = \frac{\alpha}{1 - \alpha} \frac{x_{12}^*}{x_{11}^*} = \frac{\beta}{1 - \beta} \frac{x_{22}^*}{x_{21}^*},$$

where the second equality must be satisfied by any interior Pareto optimal allocation. Thus prices in this case are well-defined. If we have a corner allocation, i.e. $(x_{i1}^*, x_{i2}^*) = (w_1, w_2)$, we just need to choose prices

$$\frac{p_1}{p_2} = \frac{\alpha}{1 - \alpha} \frac{w_2}{w_1}$$

and set the wealth of the agent getting neither good equal to zero.

The transfers that make the above prices and allocations a competitive equilibrium are

$$t_1 = p_1x_{11} + p_2x_{12} - p_1w_{11} - p_2w_{12}$$

and

$$t_2 = p_1x_{21} + p_2x_{22} - p_1w_{21} - p_2w_{22}.$$

- (b) (3 points) $u_1(x_{11}, x_{12}) = \min\{x_{11}, 2x_{12}\}$, $u_2(x_{21}, x_{22}) = x_{22}$ and $w_1 = w_2$.

ANSWER: The set of Pareto optimal allocations is the set of feasible allocations satisfying $x_{11} \geq 2x_{12}$. Otherwise, we could take away $2x_{12} - x_{11}$ from consumer 1 without reducing her utility and give it to consumer 2, making the latter strictly better off. It is easy to check that any price vector with $p_1 = 0$ and $p_2 > 0$ can support all Pareto optimal allocations. [To see it, draw an Edgeworth box.] Transfers are calculated as in (a).

- (c) (3 points) $u_1(x_{11}, x_{12}) = \max\{x_{11}, 2x_{12}\}$, $u_2(x_{21}, x_{22}) = \max\{2x_{21}, x_{22}\}$ and $w_1 = w_2$.

ANSWER: Notice that if $p_1, p_2 > 0$ both consumers will consume only one of the goods. If one of the two prices is equal to zero, the consumers will want to consume an infinite amount of one of the goods, so there is no equilibrium in this case. So the only allocation that can be supported as an equilibrium with transfers is $(x_{11}, x_{12}) = (0, w_{12} + w_{22})$, $(x_{21}, x_{22}) = (w_{11} + w_{21}, 0)$. To support this allocation we only need prices that induce consumer 1 to demand good 2 and while consumer 2 demands good 1. A sufficient condition for this to happen is

$$\frac{p_1}{2} \leq p_2 \leq 2p_1.$$

Again, the transfers are the same as in (a).

Question 3

Players 1 and 2 are involved in a dispute about which of them owns a treasure chest. Initially, the treasure chest is in the possession of player 1, but in certain circumstances player 2 may be able to sue player 1 in an attempt to take possession of the chest.

Player 1's legal claim to the chest may be strong (S) or weak (W), and player 2 may be informed (i) or uninformed (u) about the strength of player 1's legal claim. Each type of player 1 assigns probability $\frac{4}{5}$ to player 2 being uninformed. An uninformed player 2 assigns probability $\frac{1}{4}$ to player 1 having a weak legal claim, and an informed player 2 can distinguish between the two types of player 1.

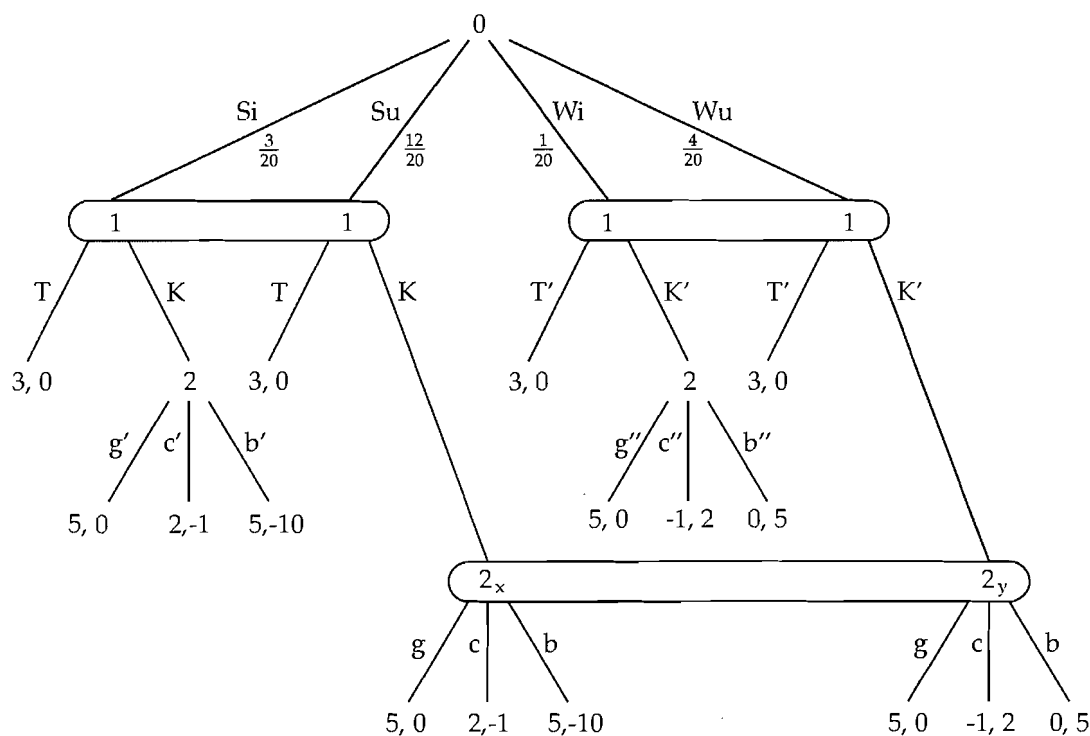
After each player's type is determined, the interaction between the players proceeds as follows: First, player 1 decides whether to trade the chest to a third party (T) or to keep the chest (K). If he trades the chest, the game ends, and the payoffs are $(3, 0)$ (= (payoff to player 1, payoff to player 2)).

If player 1 keeps the chest, then player 2 has three options: give up (g), sue cautiously (c), or sue boldly (b). If player 2 gives up, player 1 will be able to spend more time searching for a buyer, and so payoffs are $(5, 0)$. If player 2 sues cautiously, much of the surplus generated by the chest is used up in court fees; in this event, payoffs are $(-1, 2)$ if player 1 has a weak legal claim, or are $(2, -1)$ if player 1 has a strong legal claim. Finally, "suing boldly" means that player 2 attempts to obtain the chest by quasi-legal means. This attempt succeeds if player 1 has a weak legal claim (leading to payoffs $(0, 5)$), but leads to player 2's imprisonment if player 1 has a strong legal claim (leading to payoffs $(5, -10)$).

- (i) Represent the situation described above as an extensive form game. (Hint: The game tree should reflect the facts that each player learns his own type, and that a player 2 of type i learns player 1's type. It may be helpful to think of the information about own and other's type being provided in two stages. In constructing the tree, be sure to account correctly for players' knowledge at each point of play.)
- (ii) Find all of the game's sequential equilibria.
- (iii) Can forward induction be used to eliminate any of the equilibria you found? Explain.

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- (i) The game is below. Note that formally, player 2's posterior beliefs when she is of type i are determined by the beliefs of the other types. But since type i is the informed type, these beliefs are actually irrelevant.



- (ii) Clearly 2 plays g' and b'' in any sequential equilibrium, so we take this as given in the rest of the analysis. The best response correspondences for the remaining information sets are as follows:

For type $t_2 = u$, g is optimal when $\mu_2(y) \in [0, \frac{1}{3}]$, c is optimal when $\mu(y) \in [\frac{1}{3}, \frac{3}{4}]$, and b is optimal when $\mu_2(y) \in [\frac{3}{4}, 1]$. Thus g and b are never simultaneously optimal.

Type $t_1 = S$ weakly prefers T when

$$3 \geq 2 \cdot \frac{4}{5} \sigma_2(c) + 5(1 - \frac{4}{5} \sigma_2(c))$$

$$\iff \sigma_2(c) \geq \frac{5}{6}$$

Type $t_1 = W$ weakly prefers T' when

$$3 \geq -1 \cdot \frac{4}{5} \sigma_2(c) + 5 \cdot \frac{4}{5} \sigma_2(g)$$

(1) $\iff 15 + 4\sigma_2(c) \geq 20\sigma_2(g)$

These calculations imply that

- (2) if S weakly prefers T , then W strictly prefers T' ,
- (3) if W weakly prefers K' , then S strictly prefers K .

This simplifies the analysis by cases below.

Note as well that if the information set of type $t_2 = u$ is reached, her belief is

$$(4) \quad \mu_2(y) = \frac{\frac{1}{4}\sigma_1(K')}{\frac{1}{4}\sigma_1(K') + \frac{3}{4}\sigma_1(K)} = \frac{\sigma_1(K')}{\sigma_1(K') + 3\sigma_1(K)}.$$

There are two components of sequential equilibria. In the first, both types of player 1 trade for sure:

- (I) $((T, T'), c)$, with $\mu_2(y) \in [\frac{1}{3}, \frac{3}{4}]$;
- $((T, T'), \alpha c + (1 - \alpha)g)$, with $\alpha \geq \frac{5}{6}$ and $\mu_2(y) = \frac{1}{3}$;
- $((T, T'), \alpha c + (1 - \alpha)b)$, with $\alpha \geq \frac{5}{6}$ and $\mu_2(y) = \frac{3}{4}$.

In the second, both types of player 1 keep the chest, and player 2 gives up:

- (II) $((K, K'), g)$ (with $\mu_2(y) = \frac{1}{4}$).

To prove this, divide the analysis into cases according to the behavior of type $t_1 = S$. If $t_1 = S$ plays T , then $t_1 = W$ plays T' by (2). The elements of (I) are the combinations of mixed strategies for player 2 that make $t_1 = S$ prefer T , and the beliefs that support player 2's mixed strategies. Player 2's information set is unreached, and it is easy to see that any belief for player 2 is consistent. Thus, the elements of (I) above are indeed sequential equilibria.

If $t_1 = S$ mixes, then $t_1 = W$ plays T' by (2). This implies that $\mu_2(y) = 0$, which implies that player 2 plays g , which implies that both types of player 1 would like to deviate to keeping. Contradiction.

If $t_1 = S$ plays K , then (4) implies that $\mu_2(y) \leq \frac{1}{4}$, which implies that player 2 plays g . This in turn implies that both types of player 1 prefer to keep, and hence that $\mu_2(y) = \frac{1}{4}$. Thus (II) above is a sequential equilibrium.

- (iii) The only candidate for elimination by forward induction are the equilibria in component (I), in which type $t_2 = u$ is unreached. But since both $t_1 = S$ and $t_1 = W$ would benefit from deviating if player 2 were to play g , any beliefs of player 2 are reasonable, so none of the equilibria can be eliminated.

Question 4

A firm's profit may take one of two values, π_1 and $\pi_2 > \pi_1$. The firm is run by a manager, who chooses between two levels of effort: high and low. The manager has utility $U = u(w - c)$ when he exerts high effort and $U = u(w)$ when he chooses low effort, where w is the manager's wage, u is an increasing and concave function ($\lim_{w \rightarrow \infty} u'(w) = +\infty$), and $c > 0$. The manager is maximizing the expectation of u , while his reservation wage is $U_0 = u(w_0)$. The shareholders' objective function is the expectation of the net profit $\pi - w$.

If the manager exerts high effort, the profit is equal to π_2 with probability x and equal to π_1 with probability $1 - x$. If he exerts low effort, the profit is π_2 with probability y and π_1 with probability $1 - y$, $0 < y < x < 1$. The shareholders choose the manager's contract.

First, assume that the manager's effort is observed by the shareholders, who can impose the desired effort level on the manager.

- (a) Suppose that the optimal wage contract induces a low effort level. Describe the optimal contract.
- (b) Suppose that the optimal contract induces a high effort level. Describe the optimal contract.

Now, assume that the manager's effort is not observable by the shareholders.

- (c) Suppose that the optimal contract induces a high effort level. Could the contract you found in part (b) be implemented? If yes, show that it is still optimal. If not, demonstrate why and find the optimal wage structure. Compare the shareholders' profit under observable and unobservable effort.
- (d) Suppose that the optimal contract induces a low effort level. How should the shareholders induce low effort? Compare the shareholders' profit under observable and unobservable effort, and provide economic intuition for any differences between this comparison and the corresponding one from part (c).

Suppose that in the previous model, the optimal contract induces high effort when profit is verifiable. For part (e), assume that the manager effort is observable by the principal (the shareholders) but it is not verifiable, i.e., the principal cannot submit sufficient evidence to a third party (e.g., a court). Thus, contracts that depend directly on the effort cannot be made.

- (e) When profit is observable, but not verifiable, would the principal have an incentive to claim that the profit is low even if it is high? Now suppose that there is a large number of managers, so that the law of large numbers holds (in particular, if all managers exert effort, they know that x percent of them will yield high profit). As before, the probability that the individual performance is π_2 rather than π_1 is x or y , depending on whether the manager exerts high or low effort; the outcomes are independent across managers. Consider the following contract proposed by the principal: "I will pay a wage w_2 to x percent of my managers (the ones that I announce to be the most productive ones), and a wage w_1 to the rest," where w_1 and w_2 are the optimal wages under verifiability you found. Can the contract be implemented? Provide an economic interpretation. Assume that the principal does not collude with any of the managers.

Solutions sketch:

- (a) Optimal insurance requires $w_1 = w_2 = w_0$.
- (b) Optimal insurance requires a constant net wage for the manager: $w_1 - c = w_2 - c = w_0$.
- (c) A high effort cannot be induced by a constant wage structure. Instead, the shareholders must reward the manager when the profits are high. That is, the incentive constraint must be satisfied:

$$(IC) \quad xu(w_2 - c) + (1 - x)u(w_1 - c) \geq yu(w_2) + (1 - y)u(w_1).$$

Recall that $x > y$. Since the l.h.s. $xu(w_2 - c) + (1 - x)u(w_1 - c) < xu(w_2) + (1 - x)u(w_1)$, $w_2 > w_1$.

$$(IR) \quad xu(w_2 - c) + (1 - x)u(w_1 - c) \geq u(w_0).$$

The shareholders' expected profit is $x(\pi_2 - w_2) + (1 - x)(\pi_1 - w_1)$.

Both constraints must be binding: If the IC constraint were not binding, then maximization would yield full insurance ($w_1 = w_2$), which violates the IC constraint. If the IR constraint were not binding, the shareholders could reduce the wages without violating the IR constraint.

When the shareholders want to induce high effort level, their profit under unobservability is lower: by the binding IR constraint and the concavity of u , the expected wage $xw_2 + (1 - x)w_1 > w_0 + c$ (Jensen's inequality).

- (d) The wage contract under observable effort $w_1 = w_2 = w_0$ induces low effort level when effort is unobservable as well. The relative profit from inducing high effort is lower when effort is unobservable: effort, if it is not observed, must be induced through incentives. The manager's wage must grow with the realized profit.
- (e) When profit is observable, but not verifiable, the principal has an incentive to claim that the profit is low even if it is high, because $w_1 < w_2$. The principal has no incentive to misrepresent the individual performances for the proposed contract. The IC and IR constraints are satisfied: If all managers exert effort, they know that x percent of them will receive wage w_2 . Those yielding profit π_1 will receive wage w_1 . By committing himself to an overall reward structure, the principal can thus be allowed to choose a contract. Verifiability is effectively obtained.