
January 6th, 2006, from 9:00 AM to 12:30 PM.

- On the top of EACH yellow sheet, write your ASSIGNED NUMBER, date, and name of exam and question number. DO NOT write your name on the yellow pads. After the examination, the questions sheets and yellow pads will be collected. Do not write on the question sheets.

- This is a closed book exam.

- Please solve any two (2) of the four (4) problems. The time assigned to each is one hour and fifteen minutes. Thus, you have an extra hour to read the questions and revise your answers.

- The total time allotted is three and one half (31/2) hours.

- Each problem receives the same weight (100 points). The points allocated to each subsection are indicated in each problem.

- If you get stuck in a problem/section, move on. Partial credit will be granted when it is clear from your work that you were approaching the problem in a generally correct way.

- Read the problems carefully and completely before you begin your answer. The problems will not be explained. If you think that a problem is ambiguous or poorly worded, make the minimum necessary assumptions to make it beautiful and well posed.

- Please return unused portion of the yellow tablets.

- There are 9 pages in this exam (including this cover page). Please make sure that you got all of them.

- Good luck!
1 Problem I: Firm Dynamics and Equity Premium (100 points)

1.1 Firm Dynamics

Think of a firm in an industry. The output of the firm is \( q = f(\phi, n) \), where \( n \) is labor input. \( \phi \in S \equiv [0, 1] \) is a productivity shock which follows a Markov process with a conditional cumulative density function \( F(\phi' | \phi) \). A fixed cost \( c_f \) must be paid every period by the firm, as long as it stays in the industry. For given output price \( p \) and input price \( w \), let \( \pi(\phi, p, w) \), \( q(\phi, p, w) \) and \( n(\phi, p, w) \) denote, respectively, the profit, output supply, and input demand functions. Note that \( n(\phi, p, w) \geq -c_f \). The prices \( p \) and \( w \) are constant, and the firm takes them as given.

Assume that \( \pi \) is continuous and strictly increasing in \( \phi \). \( F(\phi' | \phi) \) is continuous in \( \phi \) and \( \phi' \). More importantly, it is assumed that \( F(\phi' | \phi) \) is strictly decreasing in \( \phi \)—that is, there exists a form of stochastic dominance.

The firm discounts profits with a constant factor \( 0 < \beta < 1 \), and its objective is to maximize the expected present value of its profit stream.

Each period, before the new shocks are realized, the firm decides whether to exit the industry or not. If it exits, it receives a present value which we normalize to zero, and cannot re-enter afterwards.

1. (6 points) Can we view the firm’s problem as a repetition of static profit-maximization problems? Why or why not?

2. (18 points) Formulate the firm’s problem as a dynamic programming problem. Be explicit about your choice of state variable(s) and what your value function means. Do not use the expectation operator (\( E \)), but explicitly write out the integration instead.

3. (14 points) Briefly describe what the firm’s optimal decision rule looks like. Justify your answer.

4. (8 points) Let’s assume that \( c_f = 0 \). Can we now view the firm’s problem as a repetition of static profit-maximization problems? Why or why not?

1.2 Equity Premium Puzzle with Incomplete Markets

The economy is populated with a continuum of households that have identical preferences:

\[
E_0 \sum_{t=0}^{\infty} \beta^t \frac{c_{t+1}^{1-\gamma}}{1 - \gamma}.
\]

(1)

1. (6 points) Write down the Euler equation for household \( f \) and asset \( j \) at period \( t \). More specifically, this equation relates the gross return \( R_{t+1}^{f} \) with the consumption growth rate for the consumer, \( \frac{c_{t+1}}{c_t} \).
It turns out that we have data on aggregate consumption, but not at the individual level for this economy. Nevertheless, it is known that the growth rate of individual consumption is given by:

\[
\frac{c_{t+1}^i}{c_t^i} = \frac{C_{t+1}}{C_t} \nu_{t+1}^i, \quad \log \nu_{t+1}^i \overset{iid}{\sim} \mathcal{N}\left(-\frac{\sigma_{t+1}^2}{2}, \sigma_{t+1}^2\right),
\]

where \(\frac{C_{t+1}}{C_t}\) is the growth rate of aggregate consumption. Note that \(\nu_{t+1}^i\) denotes the idiosyncratic component of consumption growth, and is i.i.d. across households once conditioned on \(\sigma_{t+1}^2\) and \(\frac{C_{t+1}}{C_t}\). \(\sigma_{t+1}^2\), \(\frac{C_{t+1}}{C_t}\), and \(R_{t+1}^i\) may well be correlated with one another.

2. (14 points) Write down the Euler equation that must hold for the aggregate consumption. The Euler equation must be written only in terms of: \(\beta\), \(\gamma\), \(\sigma_{t+1}^2\), \(\frac{C_{t+1}}{C_t}\) and \(R_{t+1}^i\). Be explicit about each step of your derivation.

Now, assume that, for some constant \(P\) and \(Q > 0\):

\[
\sigma_{t+1}^2 = P + Q (\log C_{t+1} - \log C_t).
\]

(3)

3. (6 points) What does Equation (3) tell you about the cyclicity of the idiosyncratic consumption risk? Is the idiosyncratic risk bigger in recession or in boom?

4. (6 points) Show that the Euler equation for the aggregate consumption now becomes:

\[
1 = \beta \mathbb{E}^q \left[ \left( \frac{C_{t+1}}{C_t} \right)^\gamma R_{t+1}^i \right],
\]

where \(\gamma = \gamma(1 + \gamma)\) and \(\beta = \beta \exp \left\{ \frac{\gamma(1 + \gamma)}{2} P \right\} \).

5. (22 points) How can this model help resolve the equity premium puzzle? The answer should take into account the risk-free rate puzzle.
Problem # 2: Fluctuations and Monetary Policy (100 Points)

Consider an economy populated by identical agents. Each individual has preferences defined over infinite sequences of consumption and real money balances given by

\[ U = \sum_{t=0}^{\infty} \beta^t \left[ \alpha c_t^{-\sigma} + (1-\alpha) \left( \frac{m_t}{p_t} \right)^{-\frac{1}{\rho}} \right], \]

where \( 0 < \beta < 1, 0 < \alpha < 1, \) and \( 1 + \rho > 0. \) The elasticity of substitution between consumption and real money balances is \( \sigma = (1+\rho)^{-1}. \) Each household maximizes \( U \) subject to a sequence of budget constraints given by

\[ c_t + \frac{b_{t+1}}{p_t} + \frac{m_t}{p_t} \leq y_t - \tau_t + \frac{(1+i_t)b_t}{p_t} + \frac{m_{t-1}}{p_t}, \]

with \( m_{-1} \) and \( b_0 \) given. The term \( b_{t+1} \) denotes the value (in nominal terms) of one-period bonds, \( i_t \) is the nominal interest rate, \( m_t \) is nominal money balances, \( p_t \) is the price level, \( y_t \) is the endowment of consumption, and \( \tau_t \) are taxes paid by each individual (in real terms).

The government budget constraint (on a per capita basis) is given by

\[ \tau_t + \frac{B_{t+1}}{p_t} + \frac{M_t}{p_t} = q_t + \frac{(1+i_t)B_t}{p_t} + \frac{M_{t-1}}{p_t}. \]

Let the sequence \( \{g_t\} \) be given (with \( g_t < y_t \)).

1. (10 points) Define a competitive equilibrium.

2. (20 points) Derive the demand for real money balances, \( q_t = \frac{m_t}{p_t} \) as a function of income, \( y_t \), government spending, \( g_t \), and all relevant prices and interest rates. What does the model imply about the effect of increases in \( i_{t+1-k} \) on the demand for real money balances, for \( k > 0 ? \) Explain the economic intuition behind your finding.

3. (50 points) Assume that \( y_t = y > 0 \), and that government spending satisfies

\[ g_t = \begin{cases} g_H & \text{t even} \\ g_L & \text{t odd} \end{cases} \]

To simplify, you may assume that \( g_H = g_L + v, \) \( v > 0. \) Analyze whether there exists a competitive equilibrium such that [Note: you need not worry about initial conditions; put it differently, you may assume that the initial conditions have the ‘appropriate’ values]:

(a) The nominal interest rate is constant (i.e. \( i_t = \bar{i} \)),

(b) The real value of government bonds is constant (i.e. \( B_{t+1}/p_t = B \)),

(c) Real taxes are constant (\( \tau_t = \tau \)).
4. (20 points) Assume (as before) that \( y_t = y > 0 \), and that government spending satisfies

\[
y_t = \begin{cases} 
  q_H & \text{t even} \\
  q_L & \text{t odd}
\end{cases}
\]

To simplify, you may assume that \( q_H = q_L + u, \ u > 0 \). Analyze whether there exists a competitive equilibrium such that [Note: you need not worry about initial conditions; put it differently, you may assume that the initial conditions have the ’appropriate’ values]:

(a) The nominal interest rate is constant (i.e. \( i_t = i \)),

(b) The real value of government bonds is constant (i.e. \( B_{t+1}/\pi_t = B \)).

(c) Taxes in the “low” state are 0 (i.e. \( \tau_t = \tau_L = 0 \), if \( t \) is odd), and taxes in the “high” state are constant (i.e. \( \tau_t = \tau_H \), if \( t \) is even).

If an equilibrium exists go as far as you can analyzing the impact of \( q_L/y \) upon the inflation rate.
3 Problem 3: Consumption and Adjustment Costs (100 Points)

3.1 Differentiability (30 points)
Consider the following dynamic programming problem

\[ V(a, z_t) = \max_{c, a'} \left\{ U(c) + \beta \sum_{j=1}^{n} \pi_{ij} V(a', z_j) \right\}, \]

subject to \( c + a' = z_t + (1 + r)a \),
and \( a' \geq 0 \).

Let \( U \) be a bounded, strictly increasing, strictly concave, continuously differentiable function. Suppose that \( 0 < r < \beta < 1 \). The bounded positive random variable \( z \) follows a \( n \)-point Markov process. In particular, \( z \) is drawn from the discrete set \( Z = \{ z_1, z_2, \ldots, z_n \} \) according to the probability distribution specified by \( \pi_{ij} = \text{prob}\{ a' = z_j | z = z_t \} \), where \( 0 \leq \pi_{ij} \leq 1 \), and \( \sum_{j=1}^{n} \pi_{ij} = 1 \).

a) Is the value function \( V(a, z) \) is continuously differentiable in \( a \), for all \( a > 0 \), whenever \( a' > 0 \)? If so, prove it. If not, explain what is the issue here.

b) Is the value function \( V(a, z) \) is continuously differentiable in \( a \), for all \( a > 0 \), whenever \( a' = 0 \)? If so, prove it. If not, explain what is the issue here.

3.2 Consumption (30 points)
Suppose a consumer with preferences given by

\[-\sum_{t=0}^{\infty} \beta^t e^{-\alpha c_t},\]

where \( 0 < \beta < 1 \), \( c_t \) is consumption and \( \alpha > 0 \). The consumer has initial assets \( a_0 \) and can borrow and lend at a real interest rate of \( r \) in each period. The consumer’s income in period \( t \) is \( w_t \) for \( t = 0, 1, 2, \ldots \).

a. Cast the consumer’s problem as a dynamic programming problem and derive the consumer’s Euler equation.

b. Derive an expression for the change in consumption between period \( t \) and \( t + 1 \). Also, derive an expression for the optimal choice of consumption in each period. (Your answer must depend only on primitives)

c. How is the change in consumption between \( t \) and \( t + 1 \) affected by changes in \( \alpha, \beta \) and \( r \)? Explain.
3.3 Adjustment Costs (40 points)

Consider the following problem of a firm with convex investment costs. Specifically, the total cost of investment is a function $c : \mathbb{R} \to \mathbb{R}_+$ of gross investment that is strictly increasing, strictly convex and differentiable with $c(0) = 0$. The firm operates a technology $z = f(k)$, where physical capital $k \geq 0$ is the single input and $z \geq 0$ is output. Assume that $f$ is continuously differentiable, strictly increasing and strictly concave and that

$$f(0) = 0, \lim_{k \to 0} f'(k) = +\infty, \lim_{k \to \infty} f'(k) = 0$$

Assume that $p$ is the price of output, $q$ the price of capital, and $r$ the interest rate, with $\beta = (1 + r)^{-1} \in (0,1)$. Assume that capital must be purchased one period in advance and depreciates at the rate $\delta \in (0,1)$. Then the firm's problem is

$$\sup \sum_{t=0}^{\infty} \beta^t \{pf(k_t) - c[k_{t+1} - (1-\delta)k_t]\}$$

where $\lfloor . \rfloor$ denotes the cost of gross investment (note: $\lfloor . \rfloor$ is a function, not a constant).

(a) Formulate the functional equation for this problem and show that there is a unique value function $v^*$ that solves the functional equation. Also show that $v^*$ is the supremum function for the sequence problem. Be careful to justify each step in your proof and explain why these results are applicable.

(b) Show that $v^*$ is strictly increasing and strictly concave.

(c) Show that $v^*$ is differentiable. Then use the Euler equation to show that the policy function is strictly increasing in current capital.
Problem #4 - Phillips curve

1. Write the Phillips curve as

\[ \pi_t = \gamma y_t + \beta E_t \pi_{t+1}, \]  

where \( \pi_t \) is inflation, \( y_t \) is real marginal cost, \( \gamma > 0, 0 < \beta < 1 \), \( E_t \) is expectations. Solving (1) forward and imposing a terminal condition yields

\[ \pi_t = \gamma \sum_{j=0}^{\infty} \beta^j E_{t+j} y_{t+j} \]

(2)

(10)

a. What is the rationale for (1)?

b. According to (2), should inflation rise in advance of a rise in real marginal cost and fall in advance of a fall in real marginal cost? Does (2) by itself imply anything about whether real marginal cost should rise or fall in advance of a predictable movement in inflation?

c. In light of your answer to (b), is either or both of the following Granger causality relations a logical implication of this model (i) \( \pi_t \rightarrow y_t \)  (ii) \( y_t \rightarrow \pi_t \)?

d. Suppose that the production function for the \( i \)th firm is of the form \( Q_{it} = A_i N_{it}^\theta \), for \( 0 < \theta < 1 \) and \( A_i \) and exogenous process for technology (assumed the same for all firms). Let \( W_i \) denote the nominal wage (also the same for all firms). Show that nominal marginal cost is \( (1/\theta) W_i N_{it}/Q_{it} \). If \( P_t \) is nominal price of the \( i \)th firm conclude that real marginal cost is \( (1/\theta) W_i N_{it}/P_t Q_{it} \), i.e., labor’s share.

e. Under certain approximations, economy-wide real marginal cost \( y_t \) is proportional to economy-wide labor’s share \( W_t N_t/P_t Q_t \). Using U.S. data, the dashed line in the figure below plots an estimate of the present value of labor’s share (which you may be assume to be proportional to marginal cost \( y_t \)), the solid line plots inflation. Does the plot look consistent with your answers in parts (b) and (c)?

(continued ....)

Figure 2

Fit of Labor-Share Based New-Keynesian Phillips Curve (beta=0.99)
In the rest of the question, assume that

\[(3a)\quad y_t = e_t + z_t\]
\[(3b)\quad z_t = \theta z_{t-1} + v_t,\]

where \((e_t, v_t)\) is a finite variance, zero mean i.i.d. sequence with \(E e_t v_s = 0\) all \(t, s, \) and \(|\theta| < 1\). Note that independence of \(e_t\) and \(v_t\) implies independence of \(e_t\) and \(z_t\) as well. Recall that since \(z_t\) is an \(AR(1)\), the first order autocovariance of \(z_t\) is related to the variance of \(z_t\) via \(E z_{t-1}z_t = \phi \sigma_z^2\), and the first order autocorrelation of \(z_t\) is \(\phi\).

\[(4)\quad f.\] Using \((3a)\), compute the variance and the first order autocovariance of \(y_t\), in terms of the variances and first order autocovariances of \(e_t\) and \(z_t\).

\[(16)\quad g.\] The answers to the following will often involve the model parameters \(\beta, \gamma\) and \(\phi\).
(i) What is \(E e_{t+j}e_t\) for \(j > 0\)? (Yes, this is trivial.)
(ii) What is \(\sum_{j=0}^{\infty} E e_{t+j}e_t\)? (Still trivial.)
(iii) What is \(E e_{t+j}e_t\) for \(j > 0\)?
(iv) What is \(\sum_{j=0}^{\infty} E e_{t+j}e_t\)?
(v) What is \(\sum_{t=0}^{\infty} E e_{t+j}e_t\)?
(vi) What is \(\pi_n\) in terms of current and (possibly) lagged \(e_t\)'s and \(z_t\)'s? (That is, use your answer to (v) to solve for \(\pi_n\) as a function of realized variables rather than in a form that involves expectations.)
(vii) Compute \(E \pi_n^2\) in terms of the moments of \(e_t\) and \(z_t\).
(viii) Compute \(E \pi_n \pi_{n+1}\), the first order autocovariance of \(\pi_n\), in terms of the moments of \(e_t\) and \(z_t\).

\[(10)\quad h.\] (Note: you do not need a calculator to answer this question. Feel free to express numerical answers as ratios of various numbers, even though it will be easy for many of you to reduce those ratios to decimal equivalents.) Suppose that

\[(4)\quad \sigma_e^2 = 2, \sigma_z^2 = 1, \phi = 0.9, (1-\beta \phi)^{-1} = 10, \gamma = 1.\]

(Hint: while the numbers given allow one to solve for the numerical value of \(\beta\), you will have no need to solve for this numerical value to answer the question. The only numbers you need are the ones given.)

(i) Using your answer to (f), compute the numerical value of the first order autocorrelation of \(y_t\).
(ii) Using your answers to (g)(vii) and (g)(viii), compute the numerical value of the first order autocorrelation of inflation \(\pi_t\).

\[(20)\quad i.\] In actual data, the output gap \(y_t\) is much more persistent (has much higher first order autocorrelations) than inflation \(\pi_t\). A recent literature has argued that the Phillips curve (1) and (2) therefore is inherently unappealing because, it is claimed, (2) implies that inflation should inherit the persistence of the output gap. Comment, using your answers to part (h), or your intuition if you were unable to solve part (h).