
January 5 2004

- On the top of EACH yellow sheet, write your ASSIGNED NUMBER, date, and name of exam and question number. DO NOT write your name on the yellow pads. After the examination, the question sheets and yellow pads will be collected. Do not write on the question sheets.
  - This is a closed book exam.
  - Please solve any two (2) of the four (4) problems. The time assigned to each is one hour and fifteen minutes. Thus, you have an extra hour to read the questions and revise your answers.
    - The total time allotted is three and one half (3 and 1/2) hours.
    - Each problem receives the same weight (100 points). The points allocated to each subsection are indicated in each problem.
  - If you get stuck in a problem/section, move on. Partial credit will be granted when it is clear from your work that you were approaching the problem in a generally correct way.
  - Read the problems carefully and completely before you begin your answer. The problems will not be explained. If you think that a problem is ambiguous or poorly worded, make the minimum necessary assumptions to make it beautiful and well posed.
    - Please return unused portion of the yellow tablets.
    - There are 9 pages in this exam (including this cover page). Please make sure that you got all of them.
  - Good luck!
1 Problem I: Dynamic Programming (100 points)

1. Growth with Externality (35 points): Consider an economy with infinitely many agents each with preferences given by the utility function

\[ \sum_{t=0}^{\infty} \frac{\beta^t c_t^\gamma}{\gamma} \quad 0 < \beta < 1 \quad \gamma < 1. \]

Each agent has one unit of time available in each period and can produce consumption goods according to

\[ c_t = \alpha \delta_{t+1} h_t \theta^{1-\theta} u_t. \]

where \( 0 < \theta < 1, \alpha > 0, u_t \) is the time spent in consumption goods production by the agent and \( \delta_t \) denotes the average human capital across all agents in the economy. The externality is such that each agent’s productivity increases with the amount of human capital that other agents have. An agent’s human capital evolves according to

\[ h_{t+1} = \delta h_t (1 - u_t), \]

where \( \delta > 0. \) Assume that each agent has the same initial stock of human capital.

a. Set up the social planner’s problem as a dynamic program. Note that the social planner internalizes the externality and treats \( \delta_t = h_t. \) (7 points)

b. Solve for the optimal growth rates of human capital and consumption, and solve for \( u_t. \) (8 points)

c. Set up the representative agent’s dynamic programming problem and solve for the competitive equilibrium. Also, solve for a balanced growth path of the economy. (20 points)
2. Monopolist’s problem (50 points): Consider this monopolistic version of the industry equilibrium model. A monopolist chooses output over time in order to maximize the present discounted value of its profits. Profits are defined as revenue minus cost, where the only cost that the monopolist incurs is the one associated with changing its level of production. Formally, the problem of the monopolist in sequence form is:

\[
\max_{\{y_t\}_{t=1}^{\infty}} \sum_{t=0}^{\infty} \beta^t \Pi_t,
\]

\(y_0\) given where \(0 < \beta < 1\) denotes the monopolist’s discount factor. Profits, denoted by \(\Pi_t\) and are given by

\[
\Pi_t = p_t y_t - \frac{1}{2} (y_{t+1} - y_t)^2.
\]

The price \(p_t\) is given by the inverse demand function

\[
p_t = a_0 - a_1 y_t,
\]

where \(a_0 > 0\) and \(a_1 > 1\). Notice that the market price will exclusively depend on the production decision of the monopolist. The timing of the problem is as follows: at the beginning of period \(t\) the monopolist decides on the quantity \(y_{t+1}\) it wants to sell next period (at the price \(p_{t+1}\)). Revenue at time \(t\) is collected at the beginning of the period and is determined by the production choices made in period \(t - 1\).

a. Write down this problem as a dynamic programming problem. (10 points)

b. Prove that there exists a unique value function \(V\) that satisfies the functional equation and that this value function corresponds to the supremum of present discounted utilities in the sequence problem. Be careful to specify the complete metric space within which you look for solutions to the functional equation. (20 points)

c. Write down the first order condition for the monopolists’ problem. Use it to show that the policy function \(y_t = g(y)\) is strictly increasing in \(y\). [Notice that to be able to apply the Benveniste-Scheinkman theorem and take the derivative of the value function you need to show that it is concave]. (20 points)
3. Metric Spaces (15 points): Let $X = C(a, b)$ be the set of continuous functions $[a, b] \to R$, and for $x, y \in C(a, b)$ define $\rho(x, y)$ by,

$$\rho(x, y) = \left( \int_a^b [x(t) - y(t)]^2 \, dt \right)^{\frac{1}{2}}.$$

a. Show that $(X, \rho)$ is a metric space. (5 points)

b. Is it complete? If so prove it. If not, give a counter-example. (10 points)
2 Problem II: Financing Unemployment Compensation (100 points)

Consider an economy in which unemployed individuals receive one job offer per period which is drawn from the distribution $F(w)$. (If necessary, you may assume that it has a 'nice' density $f(w)$ so that $F(w) = \int_0^w f(x)dx$.) Assume that employed individuals cannot quit but all jobs are lost with probability $\lambda$. Every individual in the economy maximizes the expected present discounted value of their labor income, with a discount factor equal to $\beta$. To make the model consistent with the steady state, we will assume that $\beta = (1 + r)^{-1}$.

There is a government that levies taxes on different sources of income (more on this later), and uses the proceeds to finance an unemployment insurance program. This program provides each unemployed person with $b$ units of consumption. The government budget constraint is then

$$bU = R,$$

where $U$ is the number of unemployed individuals, and $R$ is total revenue. Since population size plays no role in our analysis we will set it equal to 1. Thus,

$$U + N = 1,$$

where $N$ is the number of employed individuals. In what follows you are asked to study the effects of different ways of specifying the revenue function on the equilibrium unemployment rate.

1. (20 points) Consider the case in which the tax rate $\tau$ is exogenously given (this implies that the level of benefit is endogenously determined). Assume that the government taxes only employed workers. The government budget constraint is then

$$bU = N\tau \bar{w},$$

where $\bar{w}$ is the average wage rate of those who are employed. Go as far as you can describing the impact of changing $\tau$ on the long run unemployment rate.

2. (20 points) Consider the case in which the tax rate $\tau$ is exogenously given (this implies that the level of benefit is endogenously determined).
Assume that the government taxes all forms of income, and this includes unemployment compensation, at the rate $\tau$. The proceeds from this tax are used to finance unemployment benefits. In this case, the government budget constraint is

$$bU = \tau\{N\tau \bar{w} + bU\}$$

Go as far as you can describing the impact of changing $\tau$ on the long run unemployment rate.

3. (20 points) Explain the differences (if any) between your findings in sections 1 and 2.

4. (40 points) Assume now that the level of benefit $b$ is taken as exogenous (and hence $\tau$ is endogenous). The government budget constraint is as in 1. That is,

$$bU = N\tau \bar{w},$$

where $\bar{w}$ is the average wage rate of those who are employed. Argue that in general, the steady state equilibrium is not unique. Go as far as you can explaining the differences between your results in this section and those of section 1.

**Note:** In solving section 4 make sure that the choice of $b$ is consistent with the existence of a tax rate $\tau$ that is less than one. This requires ‘searching’ for equilibria in a limited range.
3 Problem III: Endogenous Technological Change (100 points)

Suppose there is a fixed amount of skilled labor, $S$, that can be used toward production of a final good $Y_t$ or for the improvement of a particular technology $A_t$, where the subscript indicates time, indexed by $t$. The proportions of skilled labor used in each activity are $S_{Yt}$ and $S_{At}$, respectively.

Technological change is modeled as an increase in the stock of $A$ according to the following expression.

$$\dot{A}_t = \sigma S_{At} A_t^\eta$$

where $\eta = 1$ and $\dot{A}_t$ denotes the derivative of $A_t$ with respect to time, which is assumed continuous.

The stock of technology $A_t$ is made up of feasible designs for different types of capital, each design is indexed by $i \in (0, A)$. Capital, skilled labor and a fixed and inelastic level of unskilled labor $L$ are used to produce $Y$. The production function of the final good $Y_t$ is given by the following equation.

$$Y_t = S_{Yt}^\alpha L^\beta \int_0^{A_t} x(i)^{1-\alpha-\beta} di.$$ 

The change in the stock of capital $K_t$ at any time $t$ is foregone consumption $C_t$; capital is assumed not to depreciate:

$$\dot{K}_t = Y_t - C_t$$

The social utility function that a social planner seeks to maximize is given by

$$\int_0^\infty U(C_t) e^{-\rho t} dt$$

$$U(C_t) = \frac{C_t^{1-\theta}}{1-\theta}$$

where $\theta \in (0, 1)$ and the rate of time discounting is $\rho > 0$.

A. Rewrite the production function for $Y_t$ in terms of $S_{Yt}$, $L$, $A_t$, $K_t$ and given parameters, making appropriate deductions regarding the relative proportions of $x(i)$ used in production. (10 points)

B. Write down the complete maximization problem of the social planner. (5 points)
C. Write the current-value Hamiltonian used to solve the maximization problem described in part B. Describe fully the necessary conditions for a solution to the maximization problem. Give a brief interpretation or intuition for each condition. (40 points)

D. Assume that a balanced growth, steady state solution to the maximization problem exists. Describe briefly what the term 'balanced growth' and 'steady state' mean in this setting. Be sure to distinguish their meanings, even though their occurrence coincides in this and other growth models. (5 points)

E. Describe the following features of the steady state balanced growth equilibrium:

i. The growth rate of $Y_t$ in terms of $S_{At}$ (10 points)

ii. The growth rate of the shadow price of $A_t$ (hint: for the growth rates of shadow prices, look back to part C) (5 points)

iii. The growth rate of the shadow price of $K_t$ (5 points)

iv. An expression for $S_{At}$, and give an economic interpretation of this and the implied steady state growth rate of $Y_t$ (10 points)

v. The role of $\eta$ in the features of this equilibrium (you may use clear math or clear prose here, as you prefer) (10 points)
Problem IV: The Empirics of Stock Prices (100 points)

Let $I_{j,t}$ denote the nominal gross rate of return on an asset that pays $1$ $j$ periods from now. According to the expectations-based theory of the term structure of interest rates,

$$I_{j,t} = \prod_{k=1}^{j} E_t \left( I_{1,t+k} \right)$$

This question will explore some implications of this theory

1. The net yield to maturity associated with $I_{j,t}$, denote this as $i_{j,t}$, is implicitly defined by the equation $(1 + i_{j,t})^t = I_{j,t}$. Using this definition, develop a description of how $i_{j,t}$ depends on current and future levels of $i_{j,t}$. Since the product of any two yields to maturity is typically small, it is permissible to approximate such products by 0. Modify your description of the yield to maturity to account for this and derive an approximation formula that expresses the yield to maturity as an additive relationship between current and future values of $i_{j,t}$. (30 points)

2. Using the approximation formula described in 2, describe how the Wold representation of $i_{j,t}$ may be used to test the expectations-based term structure theory. Work out explicit examples of the strategy for the cases where 1) $i_{j,t}$ is an AR(1) process and 2) $i_{j,t}$ is an MA(1) process. (70 points)