
January 6, 2003 from 9:00 to 12:30

- On the top of EACH yellow sheet, write your ASSIGNED NUMBER, date, and name of exam and question number. DO NOT write your name on the yellow pads. After the examination, the question sheets and yellow pads will be collected. Do not write on the question sheets.

- This is a closed book exam.

- Please solve any two (2) of the four (4) problems. The time assigned to each is one hour and fifteen minutes. Thus, you have an extra hour to read the questions and revise your answers.

- The total time allotted is three and one half (3 and 1/2) hours.

- Each problem receives the same weight (100 points). The points allocated to each subsection are indicated in each problem.

- If you get stuck in a problem/section, move on. Partial credit will be granted when it is clear from your work that you were approaching the problem in a generally correct way.

- Read the problems carefully and completely before you begin your answer. The problems will not be explained. If you think that a problem is ambiguous or poorly worded, make the minimum necessary assumptions to make it beautiful and well posed.

- Please return unused portion of the yellow tablets.

- There are 8 pages in this exam (including this cover page). Please make sure that you got all of them.

- Good luck!
1 Problem # 1: Research in Intermediate Goods and Growth (100 points)

Consider the following model. There is a single final good produced using a continuum of intermediate goods on the [0,1] interval. Time is continuous. The flow of a final good is produced using an intermediate good \( i \) at time \( t \) according to:

\[
Y_{it} = A_{it}x_{it}^\Phi
\]

with \( \Phi \in (0,1) \). Aggregate final output at time \( t \) is given by

\[
Y_t = \int_0^1 Y_{it} \, di.
\]

\( Y_t \) is the numeraire good. (That is, each variety of the intermediate good, though different from other varieties, is used in production of the same final good.)

Each intermediate good is produced one-for-one with labor. Each intermediate good sector \( i \) is monopolized by virtue of a patent.

The labor market is perfectly competitive, and labor is paid a wage \( w_t \). The aggregate labor force \( L \) is a constant.

For each intermediate good sector \( i \), there is a different research sector. Each research sector attempts to produce a new variety of an intermediate good. New varieties are assumed to have a superior technology to existing varieties, and each new variety replaces the old variety. There is free entry into each research sector. Research firms, within a given sector, compete to produce the next technology. Research firms within and across sectors work independently, though all research firms draw on the same pool of a shared technology frontier, called \( A_{it}^{\text{max}} \) at time \( t \). A new technology allows production in final good sector \( i \) at the frontier, so that \( A_{it} \) jumps discontinuously to \( A_{it}^{\text{max}} \). The research firm of sector \( i \) that successfully produces the new technology will hold a patent on that new technology and will produce the new sector \( i \)'s intermediate good, as described above. Labor is an input into research in each sector, given by \( n_{it} \). The Poisson arrival rate of new technology in sector \( i \) is \( \lambda n_{it} \), where \( \lambda \) is an exogenous given constant.

By free entry in the research sector, research labor is paid a wage equal to the expected value of research, \( \lambda V_t \). The expected value of a new technology invented at time \( t \) is \( V_t \).

Because the prospective payoff from research is equal across all sectors, \( n_{it} = n_t \). The growth rate of the technology frontier \( A_{it}^{\text{max}} \) is given by

\[
\frac{A_{it}^{\text{max}}}{A_{it}^{\text{max}}} = \lambda n_t \ln \gamma
\]

where \( \gamma > 1 \) is an exogenous constant. Define the variable \( a_{it} = \frac{A_{it}}{A_{it}^{\text{max}}} \), which has the distribution function \( H(a) = a^{\frac{1}{\gamma}} \). Define the variable \( \omega_t = \frac{n_t}{A_{it}^{\text{max}}} \).
The steady state equilibrium of the economy is defined as an equilibrium for which \( n_t = n \forall t \). In this steady state, it will also be the case that \( \omega_t = \omega \) and the growth rate of technology, \( g_t = g \forall t \).

1. (20 points) Describe the sector \( i \) intermediate good producer's profit maximization problem. Determine the production of each intermediate good \( x_{it} \) and the resulting profit for each producer.

2. (6 points) Rewrite the profit-maximizing intermediate good production, such that each sector will be identified by \( a \) rather than \( i \), using the definitions of \( a_{it} \) and \( \omega_t \). That is, rewrite \( x_{it} \) as \( \hat{x} \) as a function of \( \frac{\omega_t}{\alpha} \).

3. (10 points) Using the result of part 2. and the distribution function of \( a \), describe the condition necessary for labor market clearing at any time \( t \).

4. (20 points) Using the result of part 2. and the distribution function of \( a \), derive the steady state growth rate of aggregate final good production. Compare this to the growth rate of technology.

5. (20 points) Using the result of parts 1 and 2, determine the expression for the expected value of technology, \( V_t \) in a steady state equilibrium. Provide an explanation, in words, for the derivation of the expression. Combine this expression with the research arbitrage equation, as described in the body of the question.

6. (24 points) Using the results of part 3 and part 5, determine an expression for the allocation of labor to research. Provide, in words, a brief description of the expression and the interpretation of each of its terms. (Hint: it will be useful to make use of the fact that the profit of the intermediate good producer is proportional to \( \omega_t^{\frac{1-\phi}{\phi}} \).)
2 Problem # 2: Differences in Growth Performance (100 points)

2.1 Part I (50 points)
1. Describe what features of the neoclassical (Cass-Koopmans) growth model can potentially explain the large differences between per capita growth rates in East Asia and sub-Saharan Africa since 1960.

2. Describe how a growth model with nonconvexities in the aggregate production function can potentially explain these differences.

3. Describe a growth theory that does not fall into the models described in 1 and 2 that can potentially explain these differences.

2.2 Part II (50 points)
Outline an empirical strategy for determining which of these explanations best explains the differences in growth performance between these two groups of countries. Make explicit what assumptions you are making in terms of available data.
3 Problem # 3: Dynamic Optimization (100 points)

3.1 Ergodic Sets (20 points)

Let $S = \{s_1, s_2, \ldots \}$. Fix $k$ and let the Markov Process $S^k$ be defined by the following transition probability matrix

$$
\Pi^k = \begin{bmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 0 \\
1 & 0 & 0 & \ldots & 0 \\
\end{bmatrix}
$$

1. (6 points) What are the ergodic sets of $\Pi^k$?

2. (6 points) What is the invariant measure corresponding to $\Pi^k$?

3. (8 points) Go as far as you can describing the behavior of the process ($\Pi^k$, the ergodic sets, and the invariant distribtuion) as $k \to \infty$.

Be sure to carefully explain your answer.

3.2 Metric Spaces (20 points)

1. (10 points) Let $C^1[a, b]$ be the set of all continuously differentiable functions on $[a, b] = X \subset R$, with the norm $\|f\| = \sup_{x \in X} \{|f(x)| + |f'(x)|\}$. Is it a Banach space? If so prove it. If not provide a counter-example.

2. (10 points) Now consider this set of functions with the norm $\|f\| = \sup_{x \in X} \{|f(x)|\}$. Is it a Banach space? If so prove it. If not provide a counter-example.

3.3 Dynamic Programming - Output-Employment Correlation (40 points)

Consider the following representative agent model. There is a representative consumer who has preferences given by

$$
E_0 \sum_{t=0}^{\infty} \beta^t \{ \log c_t + \alpha \log l_t \}, 0 < \beta < 1, \alpha > 0
$$

where $c_t$ is consumption, and $l_t$ is leisure. The consumer has one unit endowment of time available in each period to allocate between work and leisure. The production technology is given by

$$
y_t = z_t k_t^\gamma n_t^{1-\gamma}
$$
where \( y_t \) is output, \( k_t \) is the capital stock, \( n_t \) is employment, \( 0 < \gamma < 1 \); and \( z_t \) is an i.i.d. technology shock. There is 100% depreciation of the capital stock in each period, so the aggregate resource constraint is given by

\[
c_t + k_{t+1} = y_t.
\]

1. Solve for a competitive equilibrium by solving the social planner’s problem. That is, set up the social planner’s problem as a dynamic program, explain what the state variables and choice variables are, and determine the optimal decision rules. (25 points)

2. How does employment depend on the current capital stock and the current technology shock, \( z_t \)? Is this model capable of explaining the observed correlation between employment and output (employment and output are positively correlated; when output is high, employment tends to be high)? Explain why or why not. (15 points)

### 3.4 Consumption Paths (20 points)

A consumer has preferences given by \( \sum_{t=0}^{\infty} \beta^t \log c_t \) has initial assets \( A_0 = 0 \) and can borrow and lend at a real interest rate \( r \); which is constant forever. Suppose also that the consumer’s income is \( y > 0 \) in each period \( t = 0, 1, 2 \ldots \). The consumer must abide by his intertemporal budget constraint. Use the Euler equation from the consumer’s problem and the intertemporal budget constraint to determine the consumer’s optimal consumption path. Compare the consumer’s consumption path to her income path and explain.
4 Problem # 4: Size Distribution and Growth (100 points)

Consider an economy in which all individuals rank consumption streams according to the function

$$U = \sum_{t=0}^{\infty} \beta^t u(c_t), \quad 0 < \beta < 1,$$

where $u$ is a $C^2$ function that is bounded and strictly concave. There is a large number of individuals (effectively a continuum) and each has a skill level given by $e_t$. If an individual with skill level $e_t$ is a “manager” at a firm that uses $k$ units of capital and $n$ units of labor, firm output is

$$y_t = e_t^{\theta} F(k, n)^{1-\theta}, \quad 0 < \theta < 1,$$

where $F(k, n)$ is a $C^2$ function, which is concave, increasing in each argument, and homogeneous of degree one. If an individual is a “worker,” he/she supplies one unit of labor. Let the cross-sectional distribution of skills be given by $G$ which satisfies

$$G(0) = 0, \ G(\bar{e}) = 1, \ G'(e) = g(e) > 0, \ \text{for} \ e \in (0, \bar{e}).$$

Thus, $G(\bar{e})$ is interpreted as the fraction of the population with skill level less than or equal to $\bar{e}$. Note that if all individuals with skill level greater than or equal to $e^*$ are employed as managers, the available number of workers is $N = G(e^*)$. Thus, individuals not employed as managers are employed as workers.

Consider the problem faced by a planner that in every period has to decide how to allocate individuals (they can be managers or workers), capital (the total stock of capital must be allocated to the different firms) and workers so as to maximize the utility of the representative agent. If $Y_t$ is aggregate output at time $t$, the aggregate resource constraint is

$$C_t + K_{t+1} - (1 - \delta)K_t \leq Y_t.$$

Let $M$ be a subset of $[0, \bar{e}]$ such that if $e \in M$ then $e$ is a manager. (Of course, if $e \in M^c$ — the complement of $M$ in $[0, \bar{e}]$, $e$ is a worker.) If $k(e)$ and $n(e)$ is the amount of capital and labor allocated to a firm managed by a manager with skill level $e$ output of that firm is

$$y(e) = e^{\theta} F(k(e), n(e))^{1-\theta},$$

and aggregate output is

$$Y = \int_M y(e) dG(e) = \int_M y(e) g(e) de.$$
Given the set $M$ and an initial stock of capital $K$ a feasible allocation is a pair of functions $k(e)$ and $n(e)$ satisfying

$$
K = \int_M k(e)dG(e) = \int_M k(e)g(e)de,
$$

$$
N = \int_M n(e)dG(e) = \int_M n(e)g(e)de,
$$

$$
N = \int_{M^c} dG(e) = \int_{M^c} g(e)de.
$$

1. (20 points) State the planner’s problem.

2. (30 points) Argue that any optimal allocation is such that, at time $t$, there exist a cutoff skill level, $e^*_t$, such that $M = [e^*_t, e]$, and $M^c = [0, e^*_t]$.

3. (30 points) Assume that the optimal rule is to find a single cutoff point $e^*_t$ (even if you could not show this to be the case). Go as far as you can characterizing the functions $k(e)$ and $n(e)$.

4. (20 points) Assume that the optimal rule is to find a single cutoff point $e^*_t$ (even if you could not show this to be the case). Since the total number of firms is $1 - G(e^*_t)$, and the total number of workers is $G(e^*_t)$, the average size of a firm (when size is measured using employment) is

$$
S_t = \frac{1 - G(e^*_t)}{G(e^*_t)}.
$$

It follows that the implications of the model for the size distribution are summarized by the time path of $e^*_t$. Show that, if $F(k, n) = Ak^\alpha n^{1-\alpha}$, $e^*_t = e^*$ independently of $K_t$.

5. **Extra Credit:** Assume that the solution to the planner’s problem is such that the capital stock, $K_t$, is increasing. Let the technology be given by $F(k, n) = A[\alpha k^{-\rho} + (1 - \alpha)n^{-\rho}]^{-1/\rho}$. Go as far as you can describing the predictions of the model for the time path of $S_t$ as a function of $\rho$.

*Note:* It is not a good idea to try the extra credit until you have finished sections 1-4.