Question 1: Consumption and Asset Prices

This question is concerned with the interrelations between consumption and asset returns. Consider a representative consumer who acts to maximize expected present value of discounted lifetime utility

$$E_0 \sum_{j=0}^{\infty} \beta^j u(\mathbf{c}_{t+j})$$

where $u(\cdot)$ is increasing, concave, and differentiable. Each asset $i$ is associated with a gross real rate of return $R_{i,t+1}$ between $t$ and $t+1$. Denote the expected net real return of the asset at time $t$ as $E_t(R_{i,t+1})$. Assume expectations are rational.

Hint: Remember that for any two random variables $x$ and $y$,

$$\text{cov}(x, y) = E(xy) - E(x)E(y).$$

i. (20 points) For each asset $i$, provide a formula that relates the expected gross real return on the asset to the marginal utility of consumption at $t$ and $t+1$.

ii. (30 points) Suppose that there exists an asset $z$ such that $E_t(R_{z,t+1}) = R_{z,t+1}$. Use your answer to i. to construct a formula for $E_t(R_{i,t+1}) - R_{z,t+1}$ and provide economic intuition for your result.

iii. (10 points) Is $\text{var}(R_{z,t}) = 0$? (Notice that this is an unconditional variance). Does $E_t(R_{i,t+1}) - R_{z,t+1} = 0$ imply $R_{i,t+1} - R_{z,t+1} = 0$? Provide economic intuition for both of your answers.
iv. (10 points) Propose a test of the model which takes the form of an excess volatility test comparing the variances of two time series.

v. (10 points). Propose a test of the model which takes the form of a test of whether a time series is a martingale difference.

Suppose that instead of a representative agent we consider a set of \( K \) consumers, each of whom maximizes

\[
E_0 \sum_{j=0}^{\infty} \beta^j u(c_{k,t+j})
\]

\( k = 1, \ldots, K \). Suppose each agent possesses the same utility function \( u(c_{k,t}) = c_{k,t}^{1-\varphi} \).

vi. (10 points) How does the existence of the asset \( z \) (defined above) restrict the relationship between \( \frac{c_{k,t+1}}{c_{k,t}} \) and \( \frac{c_{k',t+1}}{c_{k',t}} \)? Provide economic intuition for the restriction.

vii. (10 points) Suppose that one introduces an additional agent into this second environment and that the utility function of this agent is \( u(c_{K+1,t}) = \lambda c_{K+1,t} \). What does the introduction of this agent to the population imply about equilibrium expected returns for the various assets in the economy? Provide economic intuition for the effect (if any).
Question 2: Monetary Policy Advice

Suppose that you are asked to advise the Federal Reserve on whether to reduce the federal funds rate by 1 percent at the next meeting of the Federal Open Market Committee. Describe in detail how you would go about constructing the recommendation. Be explicit about the models, decision criteria, and statistical methods you would employ and make sure they are discussed in the context of the question.
Question 3: Intergenerational Transfers and Ergodic Sets

A. (80 points) Consider the following three-period model of life. In first period of life an individual is born and lives as child. The agent makes no decisions at this stage of life and lives in childhood bliss. In the second period, he enters the world as a young adult. He works and has his own child. In the third period of life the actor lives as a retiree. At the end of the period he dies and leaves a bequest to his child. The child receives this bequest at the beginning of the next period. Let the utility function for a young adult be given by

$$U(c^y) + \beta E[U(c^o) + \theta V(\bullet)], \ 0 < \beta, \theta < 1,$$

where $c^y$ is his consumption when young, $c^o$ is his consumption when old. Here $V(\bullet')$ denotes the expected lifetime utility that his child will realize given that the state $\bullet'$ occurs next period. Observe that a parent weights his child utility less than his own. Let a young adult's labor income be given by $w l$. Here $w$ is the time-invariant economy-wide wage rate. The number of hours that a young agent works, or $l$, is drawn from the cumulative distribution function $L(l)$. An old agent has two sources of income. First, he will have savings (which may be negative) from when he was young. Denote these by $s'$. Second, at the beginning of the period the old agent also inherits a bequest (which may be negative) from his now deceased father. Represent this by $b'$. Savings and bequests earn interest at the time-invariant rate $r$. Savings represent intragenerational asset accumulation and bequests are intergenerational asset accumulation, so to speak. Denote that expected life utility function of for an young agent by $V(\bullet)$. It's up to you figure out what the arguments inside $V$ should be. Denote that expected life utility function of for an old agent by $J(\bullet)$. It's up to you figure out what the arguments inside $J$ should be.

a. Formulate the dynamic programming problem that faces a young agent. (10 points)

b. Formulate the dynamic programming problem that faces an old agent. (10 points)

c. Show that the two equations describing $V$ and $J$ jointly define a contraction mapping. (Hint: Think about a function $(V, J)$ with range in $R^2$.) (20 points)

d. Take the size of the each generation to be unity. Suppose that $l = 1$ for all young adults. Let output be given by the constant-returns-to-scale
production function

\[ y = F(k, l), \]

where \( k \) and \( l \) are the aggregate stocks of capital and labor in economy. Suppose that capital accumulation is described by

\[ k' = (1 - \delta)k + i, \]

where \( i \) is aggregate investment. Formulate an equilibrium for the economy under study. (20 points)

e. Suppose that \( l = 1 \) for all young agents. Deduce what the equilibrium interest rate must be? (20 points)

B. (20 points) Suppose

\[ \Pi = \begin{bmatrix} \Pi_1 & 0 \\ 0 & \Pi_2 \end{bmatrix}, \]

where \( \Pi_1 \) and \( \Pi_2 \) are \( k \times k \) and \( (l - k) \times (l - k) \) Markov matrices. Let

\[ \Pi_1 = \Pi_2 = \begin{bmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{bmatrix}. \]

a. What is (are) the ergodic sets? (10 points)
b. What is (are) the invariant distributions? (10 points)
Question 4: Endogenous Fertility

(100 points) One of the stylized facts about economic growth is that fertility rate falls with economic development. The goal of this problem is to formulate and solve an economic model (A Barro Becker type model) that is capable of capturing this stylized fact. Consider an economy where individuals live for three periods - in the first they are children, in the second they are adults and in the third, they are retirees. Assume that households bear children in the second period of their lives. Parents care about the quantity and quality of their progeny (assume that parents place the weight \( b(n) \) on the lifetime utility of their identical children, where \( n \) is the number of children the household chooses to have). Assume that parents invest in the human capital of their children and also pass on bequests to them.

a. Write down the optimization problem in recursive form. (20 points)

b. Can standard techniques (Stokey Lucas) techniques be used to show the solution to Bellman's equation? Explain. (10 points)

c. Derive the FOCs, envelope conditions and Euler equations. (20 points)

d. What assumption is required on preferences and technology to make fertility choice greater than zero and below the maximum possible fertility choice (i.e. economically meaningful)? What is the issue here. (10 points)

e. Does this model imply that economic development reduces fertility choice? What assumptions generate this result? (20 points)

f. What is the effect of a tax on human capital on fertility choice? (10 points)

g. What does this model say about fertility differences within a given country? What is generating the variation across individuals? (10 points)