
July 27, 2007, from 9:00 AM to 12:30 PM.

- On the top of EACH yellow sheet, write your ASSIGNED NUMBER, date, and name of exam and question number. DO NOT write your name on the yellow pads. After the examination, the questions sheets and yellow pads will be collected. Do not write on the question sheets.

- This is a closed book exam.

- Please solve any two (2) of the three (3) problems. The time assigned to each is one hour and fifteen minutes. Thus, you have an extra hour to read the questions and revise your answers.

- The total time allotted is three and one half (3 and 1/2) hours.

- Each problem receives the same weight (100 points). The points allocated to each subsection are indicated in each problem.

- If you get stuck in a problem/section, move on. Partial credit will be granted when it is clear from your work that you were approaching the problem in a generally correct way.

- Read the problems carefully and completely before you begin your answer. The problems will not be explained. If you think that a problem is ambiguous or poorly worded, make the minimum necessary assumptions to make it beautiful and well posed.

- Please return unused portion of the yellow tablets.

- There are 5 pages in this exam (including this cover page). Please make sure that you got all of them.

- Good luck!
1 Problem # 1: Optimal Policy (100 Points)

Consider a policymaker who wishes to minimize the unconditional variance of \( x_t \). This process obeys

\[
x_t = ax_{t-1} - u_{t-1} + \varepsilon_t
\]

where \( \varepsilon_t \) is i.i.d \( N(0, \sigma^2) \). \( u_t \) is a control variable that is restricted to follow

\[
u_t = cx_t
\]

1. (10 points) Compute the variance minimizing policy rule and describe the spectral density of \( x_t \) under the assumption that the policymaker knows all of the parameters of the process. Would the policymaker be better off if it were possible to choose from set of rules defined by \( u_t = c(L)x_t \) (\( c(L) \) one-sided)? Explain.

2. (20 points). Suppose that the policymaker is constrained to choose rules of the form \( u_t = cx_{t-1} \). What is the optimal rule in this case? Would the policymaker be better off if it were possible to choose from set of rules defined by \( u_t = c(L)x_{t-1} \)? Explain.

3. (20 points) Repeat 1 under the assumption that the policymaker wishes to minimize \( \text{var}(x_t) + \lambda \text{var}(u_t) \). Provide intuition for the effect of \( \lambda \) on the spectral density of \( x_t \) under the optimal rule.

4. (25 points). Suppose that the policymaker does not know the value of \( a \); rather that it lies in the interval \( [\bar{a} - \kappa, \bar{a} + \kappa] \). Describe the decision problem for a policymaker who chooses \( c \) to minimize the variance of \( x_t \). Suppose that \( a \) is uniformly distributed on the interval \( [\bar{a} - \kappa, \bar{a} + \kappa] \). Will the policymaker choose \( c = \bar{a} \)? Explain.

5. (25 points). Suppose that the policymaker has minimax preferences. Characterize the optimal policy under the assumption that \( a \) is uniformly distributed on the interval \( [\bar{a} - \kappa, \bar{a} + \kappa] \).

2 Problem # 2: Firm Dynamics (100 Points)

Think of a firm in a given industry. The objective of this firm is to maximize the discounted sum of its expected profit stream. Assume that future profits are discounted with the constant market interest rate, \( 1 + r, r > 0 \). Let \( c(q) \) be a cost function that satisfies:

\[
c(0) = 0, \quad c'(0) = 0, \quad c'(q) > 0, \quad c''(q) > 0, \quad \text{and} \quad \lim_{q \to \infty} c'(q) = \infty.
\]

The firm’s total cost of producing \( q_t \) units of output is given by \( c(q_t)x_t \), where \( x_t \) is a positive random variable. In particular, let \( x_t = \xi(z_t) \), where \( \xi(\cdot) \) is a
strictly increasing and continuous function with \( \lim_{z_t \to -\infty} \xi(z_t) = \alpha_L > 0 \) and \( \lim_{z_t \to \infty} \xi(z_t) = \alpha_H < \infty \). \( z_t \) follows a stochastic process given by:

\[
z_t = z_{t-1} + \epsilon_t, \quad \epsilon_t \sim i.i.d. \mathcal{N}(0, \sigma^2).
\]

When in operation, the firm chooses its level of output \( q_t \geq 0 \) before observing the current realization of \( z_t \). The actual total cost is revealed after the firm's production. From the total cost, the firm can easily figure out what the realization of \( z_t \) was—note that \( \xi(\cdot) \) is strictly increasing and we can define \( z_t = \xi^{-1}(x_t) \).

The firm’s within-period expected-profit maximization problem is:

\[
\max_{q_t} p q_t - c(q_t)E[x_t|z_{t-1}],
\]

where \( p \) is the constant product price. Once the production is done and its actual cost (and hence profit) is observed, the firm decides whether to stay in operation or to close down and exit. Upon exiting, the firm gets paid \( W \)—which is a constant, and it ceases to be the object of our analysis. If the firm decides to stay, it enters the next period \((t+1)\) with \( z_t \).

1. (20 points) Formulate the firm’s problem into a Bellman equation. In particular, the value function should embody the expected value of a firm at the beginning of a period before the production choice is made. Note that the continuation/exit decision has to be somehow represented in the Bellman equation. Be explicit about state and control variables. Denote your value function with \( V \), and write out (using integrals) your expectation operator. If you need a notation for the c.d.f. or p.d.f. of the normal distribution, use \( F \) or \( f \).

2. (30 points) Prove that a unique, bounded and continuous solution for \( V \) in your Bellman equation exists. Don’t forget to explicitly mention the space on which your functional operator \( (T) \) is defined before you invoke certain theorems/conditions.

3. (10 points) Prove that \( V \) is strictly decreasing in the state variable. I advise you to first start by showing that \( V \) is non-increasing. Even if you cannot show the ‘strict’ part, you will receive up to 9 points.

4. (10 points) Describe what the firm’s exit policy looks like. Under what circumstances will this firm exit?

So far, we have considered only one firm’s problem. Now assume that the industry starts \( t = 1 \) with 10,000 firms. No additional entry is allowed into this industry. The 10,000 firms face the same \( p, c(\cdot), \) and \( \xi(\cdot) \), and are subject to the same stochastic process governing their cost shock. The only difference across these firms is that they start with different \( z_0 \)—firms know their own \( z_0 \). Further assume that the shocks across firms are uncorrelated. Note that the number of firms in the industry will decrease over time, because there are exits but no entries. In the following, a firm’s size is measured by its output level \( (q) \).
5. (15 points) Is the following statement true, false or uncertain? Justify your answer. Recall that, in a given period, firms produce first and then make exit decisions.

"Firms' exit rate is decreasing in size. That is, smaller firms are more likely to exit than bigger ones."

6. (15 points) Is the following statement true, false or uncertain? Justify your answer.

"Conditioning on survival, the growth rate \( \log q_{t+1} - \log q_t \) is decreasing in size. That is, smaller firms tend to grow faster than bigger ones."

3 Problem # 3: Distortions and Capital Utilization (100 Points)

Consider an economy in which each individual ranks consumption sequences according to the functional

\[
\sum_{t=0}^{\infty} \beta^t u(c_t), \quad 0 < \beta < 1
\]

where the function \( u \) is increasing, differentiable and strictly concave. This economy imports investment goods. Assume that the international price of the investment good is one, and denote the domestic price by \( p_k \). In general, \( p_k \geq 1 \).

Households own the capital stock and choose how intensively to use. Let the intensity of use be denoted \( v \), where \( 0 \leq v \leq 1 \). If a household owns \( k \) units of capital and chooses to use it at intensity \( v \), it supplies \( vk \) units of capital to the market and, in this case, the depreciation rate is \( \delta(v) = v^{1+\lambda}, \lambda > 0 \), and \( 0 \leq v \leq 1 \).

In this economy there are installation costs. If \( x \) units of investment goods are allocated to the production of new capital, they produce \( G(x, a) \) units of new capital goods. Thus, if every household chooses the same intensity, the aggregate law of motion of capital is

\[
k_{t+1} = (1 - \delta(v_t))k_t + G(x_t, a),
\]

where \( a \) is a factor in fixed supply owned by the households (each owns the same amount). The function \( G \) is assumed increasing in each argument, differentiable and concave. One interpretation of \( G \) is that it represents installation costs that are necessary to make capital goods productive.

Assume that capital income is taxed at the rate \( \tau \), and that the production function is given by

\[
F(K, N) = zK^\alpha N^{1-\alpha},
\]

where \( K \) is the total amount of capital services allocated to market production and \( N \) is population.

In what follows assume that the economy is at a steady state
1. [20 points] Assume that \( G(x_t, a) = x \), and that \( \lambda \) is sufficiently high so that the solution is interior. Argue that all individuals will choose the same intensity \( v \), and that the steady state value is independent of the price of capital and the tax rate on capital income. Go as far as you can describing how these two variables affect the capital-output ratio and the investment-output ratio (measured in domestic prices).

2. [20 points] Define a competitive equilibrium in which households purchase capital goods \( (x) \) and pay installation costs. Assume that there is a price for installation services so that, from the household’s point of view, the law of motion of capital is

\[
k_{t+1} = (1 - \delta(v_t))k_t + q_t x_t.
\]

3. [35 points] Let \( G(x_t, a) = x^{1-\theta}a^\theta \), with \( 0 < \theta < 1 \). Go as far as you can describing how the domestic price of investment goods \( (p_k) \) and the tax rate on capital income \( (\tau) \) affect capital utilization. Describe how changes in those two variables affect the capital-output ratio and the investment-output ratio (measured in domestic prices). Be explicit about the assumptions that guarantee an interior solution.

4. [25 points] Assume that a foreign economist correctly measures the investment-output ratio in this economy (i.e. \( p_k \delta(v)) k/y \)). The economist first values this ratio at international prices and then it uses this measure and the U.S. depreciation rate (assume that the U.S. is such that \( p_k = 1 \) and \( \tau = 0 \)) to estimate the country’s capital-output ratio. Can you determine the bias (if any) implicit in this procedure? Can you guess what this procedure implies for measured TFP in poor (say, high \( p_k \) and high \( \tau \)) countries?