
July 31, 2003 from 9:00 to 12:30

- On the top of EACH yellow sheet, write your ASSIGNED NUMBER, date, and name of exam and question number. DO NOT write your name on the yellow pads. After the examination, the question sheets and yellow pads will be collected. Do not write on the question sheets.
- This is a closed book exam.
- Please solve any two (2) of the four (4) problems. The time assigned to each is one hour and fifteen minutes. Thus, you have an extra hour to read the questions and revise your answers.
- The total time allotted is three and one half (3 and 1/2) hours.
- Each problem receives the same weight (100 points). The points allocated to each subsection are indicated in each problem.
- If you get stuck in a problem/section, move on. Partial credit will be granted when it is clear from your work that you were approaching the problem in a generally correct way.
- Read the problems carefully and completely before you begin your answer. The problems will not be explained. If you think that a problem is ambiguous or poorly worded, make the minimum necessary assumptions to make it beautiful and well posed.
- Please return unused portion of the yellow tablets.
- There are 7 pages in this exam (including this cover page). Please make sure that you got all of them.
- Good luck!
1 Problem I: Dynamic Programming (100 points)

1. Consider the case of a planner or an economically self-sufficient household (Robinson Crusoe?). The agent faces the following problem:

\[ \max \sum_{t=0}^{\infty} \left( \frac{1}{2} \right)^t \cdot c_t \]

where \( c_t \) represents consumption and \( \frac{1}{2} \) is the discount factor. Output may be used either for consumption or capital formation. Output is produced with capital, according to \( y_t = 4k_t - k_t^2 \). Output installed as capital in period \( t \) incurs an adjustment cost equal to \( \frac{1}{2} (k_{t+1} - k_t)^2 \).

a. What is the Bellman equation associated with the above decision problem? Prove the differentiability of the value function. Outline a clear argument. (20 Points)

b. Derive a closed form solution for the value and policy functions. (25 points)

c. Calculate the steady state value of capital. (5 points)

2. Consider the following dynamic economy populated by identical consumers and by a number of perfectly competitive firms with constant returns to scale production functions. Without loss of generality set the number of consumers equal to the number of firms and assume there is a unit measure of both (i.e. there is 1 consumer and 1 firm). At the beginning of each period consumers sell labor and capital services to the firms. Then, firms produce output and sell consumption and capital goods to consumers. Trading is sequential. In each period, the following markets are open: 1) labor services; 2) capital services; 3) consumption and capital goods. Assume for simplicity that capital fully depreciates after production (depreciation rate \( \delta = 1 \)). Preferences are of the form

\[ U = \sum_{t=0}^{\infty} \beta^t u(c_t, x_t) \]

where \( \beta \in (0, 1) \) denotes the discount factor, \( c_t \) consumption at time \( t \), \( x_t \) leisure at time \( t \). The function \( u \) is differentiable, strictly concave and strictly increasing. The representative consumer is endowed in every period with one unit of time that he can use to work or enjoy leisure. At time \( t = 0 \) the
consumer is also endowed with $k_0$ units of capital. Output $Y_t$ is produced using capital services and labor according to the production function: $Y_t = F(K_t, L_t)$; where $F$ is a constant returns to scale production function. $K_t$ and $L_t$ denote the aggregate (average) capital and the aggregate (average) labor input hired by the representative firm.

(a) Write the firm’s optimization problem. (2 points)

(b) Write the consumer’s problem in recursive form. Be careful to specify the individual and aggregate state variables for this problem, as well as any other function that the consumer would need to know to be able to solve his optimization problem. (8 points)

(c) Write down the definition of a recursive competitive equilibrium for this economy. (10 points)

3. Let $S = \{s_1, s_2, \ldots\}$ and let

$$\Pi = \begin{bmatrix}
0 & 1 & 0 & 0 & \cdots \\
0 & 0 & 1 & 0 & \cdots \\
0 & 0 & 0 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}.$$

What happens to the sequence $\Pi^k$ as $k$ becomes large? What are the ergodic sets? What is the invariant distribution? What is the issue here? (15 points)

4. Let

$$C^0([a;b]) \equiv \{f : [a;b] \to R : f \text{ is continuous}\}$$

and

$$\text{dist}(f; g) \equiv \max |f(x) - g(x)| ; f; g \in C^0([a;b]).$$

Is $C^0([a;b])$ complete with respect to the $\text{dist}(f; g)$ metric? If so prove it. If not counter it. (15 points)
2 Problem II: Firm Size, Output, and Inequality (100 points)

Consider an economy in which individuals have preferences given by:

\[ U = \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\theta}}{1-\theta} \quad 0 < \beta = \frac{1}{1+\rho} < 1 \quad \theta > 0. \]

Individuals differ in terms of their levels of human capital. Assume that every individual has either 'high' or 'low' human capital. More precisely, let

\[ h_L = h > 1, \]
\[ h_H = (1+\gamma)h, \quad \gamma > 0. \]

Assume that the number of individuals with high human capital is \( N_H \), while the number of individuals with low human capital is \( N_L \). In this economy firms are of size \( n \). That is, every firm has to hire exactly \( n \) workers. (Assume that \( N_L/n \) and \( N_H/n \) are integers for all possible \( n \).) If workers with human capital \( (h_1, h_2, ..., h_n) \) are hired by a firm with \( k \) units of capital total output is given by

\[ y(h_1, h_2, ..., h_n, k) = nA(\Pi_{j=1}^{n} h_j)k^\alpha, \quad 0 < \alpha < 1. \]

It follows that firms can differ in terms of their level of capital, as well as the composition—in terms of human capital—of their workers. Let \( w(h) \) be the market wage of an individual with human capital \( h \). Let \( r \) be the rental rate on capital. It follows that profits of a firm that hires a 'bundle' of workers \( (h_1, h_2, ..., h_n) \) and \( k \) units of capital is

\[ \pi(h_1, h_2, ..., h_n, k) = nA(\Pi_{j=1}^{n} h_j)k^\alpha - rk - \sum_{j=1}^{n} w(h_j). \]

The capital stock in this economy evolves according to

\[ K_{t+1} = (1-\delta)K_t + X_t, \]

with total output satisfying

\[ C_t + X_t \leq Y_t \]

4
where capital letters denote the economy wide aggregates of consumption, $C$, investment, $X$, and capital, $K$.

1. (20 points) Define a competitive equilibrium in which households own the capital and rent it out to firms. Go as far as you can characterizing the composition of the workforce in each firm. In particular, will firms in this economy be ‘integrated’ (i.e. their workforce will include both $h_L$ and $h_H$ workers), or will they be ‘segregated’ (i.e. their workforce will be homogeneous)?

Note: To make this argument it might be helpful to set $n = 2$. An answer to this section derived using $n = 2$ will receive full credit.

2. (20 points) Characterize the steady state.

Note: Parts 3-5 assume that all economies are at their steady states.

3. (20 points) Consider two measures of income inequality, denoted by $I_1$ and $I_2$ where

$$I_1 = \frac{w(h_H)}{w(h_L)},$$
$$I_2 = \frac{\sqrt{\text{var}(w)}}{E(w)},$$

where $\text{var}(w)$ is the cross-sectional (across levels of human capital) variance of market wages, and $E(w)$ is the cross-sectional mean of wages. In both cases, use the steady state wages to compute $I_1$ and $I_2$. What does the theory say about the effect of firm size on inequality.

4. (20 points) Consider countries that differ in terms of $A$ and $n$, but are otherwise identical. Higher levels of TFP, $A$, and bigger firm sizes, $n$, correspond to richer countries. Go as far as you can computing

a. The elasticity of output with respect to $A$.

b. The elasticity of output with respect to $n$.

According to this theory (especially for $n$ large), which factor is quantitatively more important to explain differences in output per capita.

5. (20 points) It is argued that high human capital workers are more productive because they are given better (more) capital to work with. Assume that there is perfect capital mobility across countries. Go as far as you can computing the capital per worker across firms in a given country, and across countries (that differ in terms of $A$ and $n$). Are the differences, if any, related to the ‘cost of capital’? Explain.
3 Problem III: Growth (100 points)

Consider an economy with the following structure:

Population growth \( n \)
Output produced according to the function \( F \):

\[
F (K_t, e^{\phi t}N_t)
\]

where \( \phi > 0 \) is interpreted as a constant and exogenous rate of technological change, \( K_t \) is the aggregate stock of physical capital and \( N_t \) is the aggregate labor force, both indexed by time \( t \). The function \( F \) is constant returns to scale in its arguments.

Consumers are homogenous, and either consume or save their income. There is no physical capital depreciation. Time is treated as continuous.

Instantaneous utility exhibits constant relative risk aversion:

\[
u(c_t) = \begin{cases} 
\frac{(c_t)^{1-\gamma}}{1-\gamma}, & \text{if } \gamma > 0, \gamma \neq 1 \\
\ln(c_t), & \text{if } \gamma = 1.
\end{cases}
\]

where \( u(c_t) \) is instantaneous utility at time \( t \) for an individual consuming \( c \).

Intertemporal utility is given by:

\[
\int_{t=0}^{\infty} u(c_t) e^{-\delta t} dt
\]

where \( \delta \) is the rate of time preference.

(Nota the rate of savings is not assumed to be fixed here.)

1. Describe the social planner's problem. (10 points)
2. Solving the social planner's problem, derive expressions for the dynamics of consumption and capital accumulation. (25 points)
3. In examining the social planner's problem, one may derive the marginal productivity of capital per capita in a steady state equilibrium that satisfies the necessary and sufficient conditions for the optimal paths of consumption and capital. Derive and interpret this expression. (20 points)
4. Describe the steady state equilibrium properties of this model using a phase diagram and any necessary accompanying analytics. (25 points)
5. Suppose the economy is in a steady state equilibrium and \( \phi \) exogenously increases (unexpectedly and permanently). Describe, using a phase diagram and/or analytics, the adjustment of the economy to this shock. (20 points)
4 Problem IV: The Empirics of Stock Prices

(100 points)

Consider the theory that stock prices, \( P_t \), are determined by the condition

\[ P_t = \beta E(P_{t+1} \mid F_t) + D_t \]

where \( F_t \) denotes all information available at time \( t \) and where dividends \( D_t \) follow an AR(1) process

\[ D_t = \rho D_{t-1} + \xi_t \]

with

\[ E(\xi_{t+j} \mid F_t) = 0 \ \forall \ j > 0 \]

1. (15 points) Characterize the projection of \( P_t \) onto the space spanned by current and past \( D_t \)'s.
2. (15 points) Does the answer to the previous question characterize all possible processes for \( P_t \)? If not, provide a full characterization of all such processes.
3. (15 points) Describe the Granger causal relationship between stock prices and dividends implied by this model.
4. (15 points) Suppose that the observed dividend series is a mis-measured version of the true dividend series, i.e. if \( D_t^* \) is the true series, then

\[ D_t = D_t^* + \nu_t \]

where \( E(D_t \nu_{t+j}) = 0 \ \forall \ j > 0 \). Characterize the ARMA representation of \( D_t^* \) if \( D_t^* \) is AR(1).
5. (40 points) Describe two distinct approaches to testing the stock price model that has been described and explain how measurement error will affect each of the tests.