
August 1, 2002 from 9:00 to 12:30

- On the top of EACH yellow sheet, write your ASSIGNED NUMBER, date, and name of exam and question number. DO NOT write your name on the yellow pads. After the examination, the question sheets and yellow pads will be collected. Do not write on the question sheets.

- This is a closed book exam.

- Please solve any two (2) of the four (4) problems. The time assigned to each is one hour and fifteen minutes. Thus, you have an extra hour to read the questions and revise your answers.

- The total time allotted is three and one half (3 1/2) hours.

- Each problem receives the same weight (100 points). The points allocated to each subsection are indicated in each problem.

- If you get stuck in a problem/section, move on. Partial credit will be granted when it is clear from your work that you were approaching the problem in a generally correct way.

- Read the problems carefully and completely before you begin your answer. The problems will not be explained. If you think that a problem is ambiguous or poorly worded, make the minimum necessary assumptions to make it beautiful and well posed.

- Please return unused portion of the yellow tablets.

- There are 8 pages in this exam (including this cover page). Please make sure that you got all of them.

- Good luck!
1 Problem # 1: Job Matching with Endogenous Firing (100 points)

Consider an economy in which when a worker “meets” a firm he/she draws a productivity level \( z \in [0, \bar{z}] \). The variable \( z \) has cdf \( F(z) \). The realization of the productivity shock is constant for as long as the worker and the firm remain matched. If a match has productivity \( z \), the firm’s profit in that period is \((1 - \tau)z - T\), where \( \tau \) is a proportional corporate income tax, and \( T \) is a lump sum tax paid by firms. A newly formed match draws its productivity from \( F(z) \).

In every period a firm who is matched with a worker can either retain its worker or fire the worker and search for another. If the firm fires the worker, it finds a new worker with some probability and, after finding it, they jointly draw a productivity level. Thus, in each period there are four possible scenarios:

1. The firm retains its current worker, generates a profit equal to \((1 - \tau)z - T\) and the worker stays with the firm next period. (The firm “learns” if the worker will stay or leave at the end of the current period, after production takes place.) The worker stays with the firm with probability \( p \).

2. The firm retains its current worker, generates a profit equal to \((1 - \tau)z - T\) and the worker leaves the firm next period. (The firm “learns” if the worker will stay or leave at the end of the current period, after production takes place.) The worker leaves the firm with probability \((1 - p)\).

3. The firm fires the current worker. It generates a profit equal to \(-T\) this period. Next period it finds a new worker (this happens with probability \((1 - q)\)). Given that a worker is found, the productivity level is drawn from \( F(z) \). (The realization is not known until next period.)

4. The firm fires the current worker. It generates a profit equal to \(-T\) this period. Next period it does not find a worker (this happens with probability \( q \)).

The objective of the firm is to maximize the expected discounted value of profits. The discount rate is \( \beta \in (0, 1) \).

a) (20 points) Derive the Bellman equation that describes the value of the firm. Argue that the optimal policy is of the “reservation productivity” variety; i.e. all workers with productivity less than a certain threshold are fired.

b) (20 points) What is the effect of an increase in the (exogenous) rate of retention \( p \), and the rate at which new workers are found \( q \) on the reservation productivity? Show your work.

c) (20 points) What is the effect of a mean preserving spread on the distribution of skills, \( F(z) \), on the reservation productivity? Show your work.
d) (20 points) A government economist argues that an increase in the lump-sum component of the tax system \((T)\) will make having a vacant job more costly and, hence, promote employment (i.e. result in a lower productivity threshold). He also argues that an increase in the proportional tax \((\tau)\) will lower the gains from finding a high \(z\) worker and, consequently, will also result in more retention (i.e. a lower productivity threshold). Discuss these claims as carefully as you can.

e) (20 points) How would your answer to e) change if the tax code is such that firms who had vacant jobs paid a different lump-sum tax. More precisely, let the after tax profit of a firm matched to a worker be \((1 - \tau)z - T_R\), while the after tax profits of a vacant firm is \(-T_V\). Discuss (and interpret) as thoroughly as you can the effects that the tax code — as defined by the triplet \((\tau, T_R, T_V)\) — has on the equilibrium productivity of the marginal worker. Explain the differences (if any) between your findings in this section and those in section d).
2 Problem # 2: Solow vs. Increasing Returns to Scale (100 points)

Assume that one has data on per capita output, human and physical capital saving rates, population growth rates, as well as measures of tariffs, public infrastructure, political regime, and income inequality, for a panel of countries. You wish to evaluate whether aggregate economic growth is better described by the Solow growth model or by a new growth alternative in which aggregate output exhibits increasing returns to scale; you also must make a policy evaluation. Add whatever assumptions you deem appropriate to allow the following questions to be precisely and formally answered.

a) (30 points) Formulate a statistical description of cross-country growth data based on the Solow model as well as one based on a new growth alternative. Be sure to be precise in your choice of a new growth alternative.

b) (30 points) Describe how one would discriminate between the two models based on the available data set.

c) (40 points) Suppose you are a policymaker who is deciding whether to shift some government expenditures from defense spending (defense expenditures are assumed to have no direct growth implications) to human capital formation. Formulate a decision problem for this and describe how the information in b) is relevant.
3 Problem # 3: Taxes, Voting and Growth (100 points)

Consider an economy with the aggregate production function

$$y(t) = Ak(t)^a g(t)^{1-a} l(t)^{1-a}$$

where $0 < a < 1$, $y$ is aggregate production, $A$ is a constant productivity parameter, $k$ is the aggregate stock of capital, $g$ is aggregate government spending on public goods, and $l$ is the aggregate stock of unskilled labor. Capital is interpreted broadly, to include notions of human capital and proprietary technology, as well as physical capital. Aggregate output, $y$, can be used for consumption or investment, and is the numeraire good. Time is indexed by $t$, and time is continuous.

To finance spending on public goods, a government presiding over this economy can tax capital in the economy at rate $\tau$. The government’s budget is balanced at each instant of time, so that

$$g(t) = \tau(t) k(t).$$

There is no expropriation of capital by the government.

There is perfect competition in factor markets. Labor is paid a wage $w(t)$ and capital earns a rate of return $r(t)$. Labor is supplied inelastically, and the aggregate stock of labor $l(t)$ is set to the value 1.

Individuals in this economy supply labor, own and supply capital, and consume. They are infinitely-lived and the population is constant in size. Individuals differ only in their initial ownership of $k(t)$ and $l(t)$. Individuals are indexed by $i$. Define the relative factor endowment of each individual $i$ by

$$\sigma^i(t) = \frac{\bar{k}^i(t)}{k(t)}, \quad \sigma^i(t) \in [0, \infty)$$

where $\bar{k}^i(t)$ and $k^i(t)$ denote the factor endowments of individual $i$ at time $t$.

Utility of all individuals follows the same structure, whereby individuals choose a path of consumption to maximize utility:

$$\max_{C^i(t)} U^i = \int_0^\infty \log c^i(t) e^{-\rho t} dt$$

where $c^i(t)$ is individual $i$’s consumption at time $t$ and $\rho$ is the rate of time preference. An individual’s income $y^i(t)$ can be used toward consumption or capital accumulation. The consumer takes the paths of $r(t), k(t)$ and $\tau(t)$ as given, where $r(t)$ is the return to capital.

a) (5 points) Derive expressions for the wage, $w(t)$, and $r(t)$ as functions of $\tau(t)$. Derive an expression for the budget constraint of a consumer $i$ in terms of individual total income $y^i(t)$, as a function of $\sigma^i(t)$ and $k^i(t)$.
b) (12 points) Solve an individual i’s utility maximization problem to derive the growth rate of \( c^i(t) \) in terms of \( k(t) \), \( \tau(t) \), and \( \rho \).

c) (Total 23 points)

1. Assuming that \( \tau(t) \) is constant, derive an expression for level of consumption \( c^i(t) \) of individual i at any time t along a steady-state path for which the growth rates of \( c^i(t) \), \( k^i(t) \) and \( y^i(t) \) are equal. (3 points)

2. Describe the dependence, along this steady-state path, of the growth rate of \( y^i(t) \) on the initial distribution of endowments \( k^i(t) \) and \( l^i(t) \). (5 points)

3. Describe the path of \( \sigma^i(t) \). (5 points)

4. Solve for the growth-maximizing tax rate \( \tau^* \). Describe how the steady-state growth of \( y^i(t) \) varies with the permanent change in the tax rate \( \tau \). (10 points)

d) (20 points) Consider a government’s choice of the optimal tax rate for individual i, \( \tau^i \), along a steady-state growth path. Using your results from part C, write down the government’s problem and solve for \( \tau^i \). (Hints: Be careful to include all appropriate constraints, including the economy-wide growth of \( k(t) \) since it enters into \( \sigma^i(t) \). Also, the tax rate derived should be time-invariant, consistent with the assumption of part c.)

e) (10 points) Using your results from part d), describe the optimal tax rate of a person with no labor income \( l^i = 0 \) at t, and the resulting growth rate if chosen by the government, relative to that implied by \( \tau^* \). Do the same for the optimal tax rate of a person with \( l^i > 0 \).

f) (Total: 30 points) Now suppose that the tax rate is chosen by simple majority rule. Assume that the median voter theorem can be applied. Define \( \sigma^m \) as the relative factor endowment share of the median voter.

1. Provide a complete statement of the median voter theorem. (Note: You do NOT need to prove the theorem.) (10 points)

2. Using your results from parts c) and d), describe the tax rate chosen by the median voter. (5 points)

3. Provide an economic interpretation for that \( \sigma^i = 1 \) across all individuals i in the economy. Provide an intuition for \( [\sigma^m - 1] \) as an index of inequality, where \( \sigma^m \) indexes the median voter \( m \). Using this index of inequality, describe the implications of the model for the relationship between inequality and growth. (15 points)
4 Problem # 4: Dynamic Optimization (100 points)

This question has three parts. Please answer all of them.

a) (30 points) Optimal Portfolio: Consider the following Bellman Equation:

\[ V(k) = \max_s \left\{ \frac{(k - s)^{1-\sigma}}{1-\sigma} + \beta E[V(k')] \right\}, \]

where \( k' = s\tilde{R}_p, \sigma > 0, \sigma \neq 1, 0 < \beta < 1, \tilde{R}_p = \omega \tilde{R} + (1-\omega)R \) is constrained to be strictly positive, \( \tilde{R} \) is a strictly positive random return, and \( R \) is a strictly positive risk-free return.

(i) Derive a closed-form solution for the value function \( V(k) \). (10 points)
(ii) Determine the policy rules, \( s(k) \) and \( \omega(k) \). (10 points)
(iii) Determine the condition on \( R \) and \( \tilde{R} \) under which it is optimal to invest a strictly positive amount \( \omega \) in the risky asset. (10 points)

b) (20 points) Metric Spaces: Let \( X \) be any non-empty set and for \( x, y \in X \), define \( d(x, y) \) by

\[ d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases} \]

Then \( (X, d) \) is called a discrete space. Is a discrete space a metric space? If so, prove it. If not, provide a counter-example. (20 points)

c) (50 points) Growth: Consider an economy where a single homogeneous good is traded. The good can be used for consumption or for investment. The economy is populated by a large number of identical agents. Life-time utility of a representative agent is given by

\[ E_0 \sum_{t=0}^{\infty} \beta^t [\ln C_t + \gamma (1 - N_t)], \ 0 < \beta < 1 \]

where \( N_t \) is the time spent working. Output is produced according to \( Y_t = A_t K_t^\alpha N_t^{1-\alpha}, 0 < \alpha < 1 \). \( A_t \) is a stochastic productivity shock. Assume that

\[ \ln A_t = \rho \ln A_{t-1} + \varepsilon_t, \varepsilon_t \sim iid N(0, \sigma_\varepsilon^2). \]

Further assume that \( \varepsilon_t \) is observed before any period-\( t \) decisions are undertaken. Physical capital depreciates at rate \( \delta \).

(i) What is the Bellman equation that corresponds to the above problem. What are the states and controls? (10 points)
(ii) Derive the Euler equation and the first order conditions for the fraction of time spent. (10 points)
(iii) One of the ‘stylized facts’ over the last hundred years in the United States is that while GDP per capita has been growing at a constant rate, fraction of life-time spent working has been decreasing. Now assume that $\delta = 1$. Is there any hope for this model to match the above ‘fact’? In answering this question you could alter the manner in which productivity evolves. Explain and account for it clearly. (20 points)

(iv) Now, assume that $\gamma = 0$ and $A_t = A$ for all $t$. If $\delta < 1$, can the value function take the form $V(K) = E + F\log(K)$ for constants $E$ and $F$? Why? (10 points)