1 Problem # 1: Taxes in the Steady State (100 points)

Consider an economy populated by a large number of identical households. Each individual solves the following problem,

$$
\max_{t=0}^{\infty} \beta^t u(c_t, 1 - n_{mt} - n_{ht}) \quad 0 < \beta = \frac{1}{1 + \rho} < 1
$$

subject to,

\begin{align*}
(1 + \tau^c) c_t + (1 + \tau^x) x_t + b_{t+1} & \leq r_t k_t - \tau^k (r_t - \delta_h) k_t - T_t + (1 - \tau^n) w_t x_t + (1 + (1 - \tau^b) r_t) b_t & (2a) \\
(1 - \delta_k) k_{t+1} & \leq k_t + x_t & (2b) \\
h_{t+1} & \leq (1 - \delta_h) h_t + B n_{mt} h_t^{1-\psi} & (2c) \\
z_t & \leq Z n_{mt} h_t^{1-\psi} & (2d)
\end{align*}

where \( c_t \) is consumption at time \( t \), \( n_{mt} \) is hours allocated to market activities, \( n_{ht} \) is hours allocated to the production of human capital, \( x_t \) is investment in physical capital, \( k_t \) is the stock of physical capital available at the beginning of time \( t \), \( h_t \) is the stock of human capital available at the beginning of period \( t \), \( b_t \) is the stock of government bonds held at the beginning of time \( t \), \( z_t \) is the supply of “effective” labor, and \( T_t \) are transfers to the government (they could be negative). In addition to the various prices (\( r_t \) is the rental price of capital, \( w_t \) is wage rate per unit of effective labor and \( r_t^b \) is the interest rate on government bonds) there are a number of taxes levied by the government: \( \tau^c \) is a tax on consumption, \( \tau^x \) is the tax rate on purchases of investment goods, \( \tau^k \) is the tax rate on capital income (minus depreciation allowances)\( [(r_t - \delta_h) k_t] \), \( \tau^n \) is the tax rate on labor income, and \( \tau^b \) is the tax rate on interest income on government bonds.

Embedded in this formulation is the assumption that firms do not care about effort or human capital separately. All they care about is how many units of effective labor the household delivers. Thus, \( w_t \) is interpreted as wage rate per unit of effective labor. Hourly wages for an individual with human capital \( h_t \) and who supplies \( n_{mt} \) hours to the market are just \( w_t z_t / n_{mt} = w_t Z n_{mt} h_t^{1-\psi} / n_{mt} \).

The function \( u \) is assumed to be twice differentiable and concave. Let \( \ell \equiv 1 - n_{mt} - n_{ht} \). Then, the marginal rate of substitution is \( \Phi(c, \ell) \), where \( \Phi(c, \ell) = u_{c}(c, \ell)/u_{\ell}(c, \ell) \). Assume

\textbf{Condition 1 (MRS)} The function \( \Phi(c, \ell) \) is increasing in \( c \) and decreasing in \( \ell \).

There is a large number of identical firms that purchase capital and “effective” labor in spot markets. Firms maximize profits given by,

$$
\Pi_t = \max_{k_t, h_t} F(k_t, z_t) - r_t k_t - w_t z_t,
$$
where the function $F$ is assumed to be twice differentiable and concave.

Finally, aggregate feasibility requires that,

$$c_t + x_t + g \leq F(k_t, z_t).$$

1. (10 points) Define a competitive equilibrium.

2. (10 points) Define a steady state and present a set of conditions (equations) whose solution is the steady state.

3. (10 points) Discuss the following claim: Given any steady state allocation associated with tax rates $(\tau^k, \tau^z)$, it is always possible to support the same allocation with tax rates $(\tilde{\tau}^k, 0)$, for some $\tilde{\tau}^k$.

4. (10 points) Discuss the following claim: In the steady state, changes in $\tau^b$ have no effects on neither the steady state allocation nor on the amount of government revenue.

5. (30 points) Go as far as you can determining the effect of an increase in $\tau^n$ (compensated by a change in transfers) on the steady state allocation. In particular, indicate what happens to the stocks of physical and human capital, and to output.

6. (30 points) Go as far as you can determining the effect of an increase in $\tau^k$ (compensated by a change in transfers) on the steady state allocation. In particular, indicate what happens to the stocks of physical and human capital, and to output.

Remark 2 If in parts 5 and 6 you cannot sign the effects on the allocation of the indicated tax changes in the general case, consider an economy in which the production function is of the CES variety: $F(k, z) = A \left[ \alpha k^{-\theta} + (1 - \alpha)z^{-\theta} \right]^{-1/\theta}$, where $1/(1 + \theta)$ is the elasticity of substitution between capital and labor which is restricted to be between 0 (perfect complements) and $\infty$ (perfect substitutes). In this setting consider the the two extreme cases of $\theta = -1$ and $\theta \rightarrow \infty$. 

3
2 Problem #2: Endogenous Technological Change
(100 points)

Consider an economy with endogenous technological change. There is free entry into a sector that produces R&D, according to a production function specified below.

Each innovation is a blueprint for producing a new variety of an intermediate good, $x_{it}$, where $i$ indexes the variety of intermediate good and $t$ indexes time. Each innovation is made excludable by an infinitely-lived patent. Each variety of intermediate good $x_{it}$ is produced by a profit-maximizing local monopolist who purchases blueprints for the good from the R&D sector firm that produced the innovation. The price of an innovation is denoted by $P_{At}$.

Intermediate goods are produced via a one-for-one technology using a capital good $K_t$ at marginal cost $r$. An intermediate good can be costlessly transformed back into that capital good. The final goods sector is perfectly competitive. Time is continuous.

To close the model, the consumption decision is characterized by a representative consumer who maximizes an additively separable utility function subject to the dynamic budget constraint. She maximizes

$$\max_{c_t} \int_0^\infty e^{-r t} u(c_t) \, dt$$

subject to

$$\dot{K}_t = r K_t + w_t L_{Y_t} + w_t L_{At} - P_{At} \dot{A}_t + A_t \pi_{xt} - C_t$$

where $s = \frac{\partial x_{it}}{\partial t}$, $w_t$ is the wage level, $\dot{A}_t$ is the rate of technological change at instant $t$, $A_t$ is the level of technology at time $t$, $\pi_{xt}$ is the profit of each intermediate good producer at time $t$, and $C_t$ is aggregate consumption at time $t$. Per capita income is $c_t$; assume the utility function $u(\cdot)$ exhibits constant relative risk aversion of $\sigma$.

1. (40 points) There is a stock of labor $L_t$ that grows at an exogenous rate $n > 0$. Labor can be used in final good production ($L_{Y_t}$) or in research ($L_{At}$). Aggregate final goods production is given by

$$Y_t = L_{Y_t}^{1-\Psi} \int_0^{A_t} x_{it}^\Psi \, di, \quad 0 < \Psi < 1.$$ 

Research sector production is given by

$$\dot{A}(t) = \delta A_t^\Phi L_{At}^\Lambda$$

with $\Phi, \Lambda > 0$. Solve the model for a decentralized economy to derive a steady state growth rate in which prices and quantities are such that all markets clear. These markets are the labor market, the intermediate good market, the innovation market, and the final good market.
2. (40 points) Now reconsider a model where \( L \) is unskilled labor, and \( H_t \) is skilled labor. Skilled labor \( H_t \) is used in both final good production (\( H_Y \)) and in research (\( H_A \)). Unskilled labor is used only in final goods production. Both are fixed exogenously, \( n = 0 \). Aggregate final goods production is given by

\[
Y_t = H_Y^{1-\Psi} \frac{n}{L} \int_0^{A_t} x_{it} \Psi dt, \quad 0 < \Psi < 1.
\]

Research sector production is given by

\[
A(t) = \delta A_t H_A.
\]

Solve the model for a decentralized economy to derive a steady state growth rate in which prices and quantities are such that all markets clear. These markets are the labor market, the intermediate good market, the innovation market, and the final good market. (You should do not need to repeat analysis done in Part 1. but should refer the examiner to the relevant result of your Part 1 answer.)

3. (20 points) Briefly comment on and compare your results from Parts 1 and 2. You may focus specifically on the functional form of research production in each model, and on the assumptions regarding population growth in each model. Might different assumptions change the results in important ways?
3 Problem # 3: Dynamics (100 points)

There are four parts to this question. Answer all four

3.1 Part 1 (30 points)
Consider the following dynamic programming problem:

\[ V(a, z_i) = \max \{ U(c) + \beta \sum_{j=1}^{n} \pi_{ij} V(a', z_j) \} , \]

subject to

\[ c + a' = z_i + (1 + \tau) a \]

and

\[ a' \geq 0 \]

Let \( U \) be bounded, strictly increasing, strictly concave, continuously differentiable function. Suppose that \( 0 < \tau < \beta < 1 \). The bounded positive random variable \( z \) follows an \( n \)-point Markov process. In particular, \( z \) is drawn from the discrete set \( Z \equiv \{ z_1, z_2, \ldots, z_n \} \) according to the probability distribution specified by

\[ \pi_{ij} = \text{prob}(z' = z_j \mid z = z_i) , \]

where \( 0 \leq \pi_{ij} \leq 1 \) and \( \sum_{j=1}^{n} \pi_{ij} = 1 \).

a) Is the value function \( V(a, z) \) continuously differentiable in \( a \), for all \( a > 0 \), whenever \( a' > 0 \)?

b) Is the value function \( V(a, z) \) continuously differentiable in \( a \), for all \( a > 0 \), whenever \( a' = 0 \)? What is the issue here?

3.2 Part 2 (15 points)
Write down the Bellman equation for the following problem: Consider the problem of a firm which faces a firm specific productivity shock \( s_t \) each period. The price \( p \) for the firm’s product and wage rate \( w \) are constant over time. Output is a function of employment \( n_t \) and \( s_t \) and given by \( f(n_t, s_t) \). Upon observing \( s_t \), firm decides how much labor to employ for the current period. Changing the level of employment implies an adjustment cost that is given by \( g(n_t, n_{t-1}) \). The shocks that the firm faces next period depends on the value of the shock in the current period. The firm also has the option to exit in the next period.

3.3 Part 3 (15 points)
Consider the following functional equation,

\[ (Tw)(x) = \beta \int f(x')dF(x', x) + g(x) . \]

a) Let \( C(X) \) be the set of all bounded and continuous functions on \( X \subseteq \mathbb{R}^d \). Under what conditions on \( F(\cdot, x) \) and \( g(x) \), is \( T : C(X) \rightarrow C(X) \)?

b) Show that \( T \) is a contraction.
3.4 Part 4 (40 points)

Consider the following two-sector economy. Consumption goods are produced using the following technology:

\[ c_t = k_t^{\beta} n_{1t}^{1-\beta}, 0 < \beta < 1 \]

Capital goods depreciate completely in one period and are produced using labor only:

\[ k_{t+1} = n_{2t}. \]

The total amount of labor in this economy is 1, so the following condition must be satisfied:

\[ n_{1t} + n_{2t} = 1 \]

Let preferences be given by

\[ \sum_{t=0}^{\infty} \beta^t u(c_t) \]

where \( u : R_+ \to R \) is strictly concave, strictly increasing and twice continuously differentiable.

(a) (5) Write down the recursive representation of the above sequence problem.

(b) (5) Show that

\[ g(0) = 1. \]

(c) (10) Show that, for all interior \( k \), \( g(k) \) - the policy function - satisfies

\[ F_2(k, g(k)) + \beta V_1(g(k)) = 0. \]

(d) (20) Assume \( u(c) = c^\gamma / \gamma \), where \( 0 < \gamma < 1 \): Show formally that \( g(k) \) falls as \( k \) increases. In the simple growth model discussed in class (i.e., the one in which \( c + k' \leq k^\beta + (1 - \delta)k \)), \( g(k) \) increases as \( k \) increases. Give the mathematical reason for the difference in results. Explain the economic intuition underlying the difference.
4 Problem # 4: Trade and Interest Rates (100 points)

Assume that economy $i$ is populated by a large number of identical individuals with utility functions given by,

$$U = E \left[ \sum_{t=0}^{\infty} \beta^t \left( \frac{c_{i1t}^{\alpha} c_{i2t}^{1-\alpha} \ln \theta}{1-\theta} \right) \right] \quad 0 < \beta, \alpha < 1 \quad \theta > 0.$$

In this economy each individual owns one tree that drops dividends (fruit) given by $c_i^t = (e_i^t, 1)$ of goods one and two. Assume that the price of good two is set equal to one (it is the numeraire) and the price of good one, $p_{i1t}$, adjusts to make supply equal to demand. The assumption about the numeraire means that all financial assets—equity on trees and interest rates—are denominated in terms of good two.

Assume that $e_{it} = z_i^t \bar{e}_t$, and that $z_i^t$ and $\bar{e}_t$ are independent, over time and from each other, random variables. Moreover, $\ln z_i^t \sim N(\mu_x, \sigma_x^2)$, and $\ln \bar{e}_t \sim N(\mu_\epsilon, \sigma_\epsilon^2)$. In other words, $z_i^t$ and $\bar{e}_t$ are independent log-normal random variables. Since it is convenient to impose $E[ z_i^t] = 1$, we set $\mu_x = -\sigma_x^2/2$. For future reference we note that countries differ only in the realization of $z_i^t$. That is, all countries share the same realization of $\ln \bar{e}_t$, but they get their “own” draw of $\ln z_i^t$ which is drawn from the common distribution $N(\mu_x, \sigma_x^2)$.

1. (10 points) Define an autarkic (no trade between countries) equilibrium in which shares in trees and one period bonds (denominated in units of good 2) are traded.

2. (15 points) Go as far as you can deriving an expression for the one period interest rate in country $i$ in autarky (i.e. no trade), $R_{iA}$.

3. (40 points) Assume now that there is a large number of countries and that they can trade goods—but not trees—in every period. Suppose that $\frac{1}{N} \sum_{i=1}^{N} z_i^t = 1$ (i.e. that the law of large numbers hold), and that good two is the international numeraire (i.e. its price is equal to one). There is no international borrowing and lending. That is, at the common international price for good one, $p_{i1t}^*$, each country’s expenditure on both goods, $p_{i1}^* c_{i1}^t + c_{i2}^t$, must equal the value of its fruit output, $p_{i1}^* e_i^t + 1$. Go as far as you can deriving an expression for the one period interest rate in country $i$ in this setting (i.e. restricted trade), $R_{iF}$. Is the rate common to all countries?

4. (15 points) Compare the variances of $\ln R_{iA}$ and $\ln R_{iF}$. Is it possible that partial trade liberalization results in an increase in the volatility of the log of interest rates relative to autarky ($\text{Var}[\ln R_{iF}] > \text{Var}[\ln R_{iA}]$)?

5. (20 points) Assume that the government of economy $i$ imposes a tax on purchases of good 1. Thus, the domestic price of good one is $p_{i1t}(1 + \tau^i)$,
where we assume that this policy change does not affect the international price found in part 3. Describe the impact, if any, of this trade restriction on the one period interest rate prevailing in country $i$.

Remark 3 If a variable $x$ is log-normal $N(\mu, \sigma^2)$, then $E[x] = e^{\mu + \frac{\sigma^2}{2}}$, and $Var[x] = e^{(2\mu + \sigma^2)} e^{(\sigma^2 - 1)}$. 