University of Wisconsin
Microeconomics Prelim Exam with Solution Sketches
Monday, June 9, 2014: 9AM - 2PM

• There are four parts to the exam. All four parts have equal weight.
• Answer all questions. No questions are optional.
• Hand in 12 pages, written on only one side.
• Write your answers for different parts on different pages. So do not write your answers for questions in different parts on the same page.
• Please place a completed label on the top right corner of each page you hand in. On it, write your assigned number, and the part of the exam you are answering (I,II,III,IV). Do not write your name anywhere on your answer sheets!
• Show your work, briefly justifying your claims. Some solutions might be faster done by drawing a suitable diagram.
• You cannot use notes, books, calculators, electronic devices, or consultation with anyone else except the proctor.
• Please return any unused portions of yellow tablets and question sheets.
• There are five pages on this exam, including this one. Make sure you have all of them.
• Best wishes!
Slippery Pete must apply to college. There are two colleges in choice set, called 1 and 2. Each costs \( c > 0 \) to apply to. To get accepted at and attend college \( i \) yields \( u_i \) utils, where \( 0 < u_2 < u_1 \). Assume that an application to college \( i \) is successful with chance \( \alpha_i \), where \( 0 < \alpha_1 < \alpha_2 < 1 \), and that students maximize expected college utility minus costs. Assume that Slippery Pete applies to neither, or one, or both colleges in period 1 only, and then attends the best college he gets into, if any. Characterize Slippery Pete’s optimal college application strategy in \((\alpha_1, \alpha_2)\)-space for each of the following two cases:

1. Admission at the two colleges are independent events.

2. Admissions at the two colleges are not independent events. Instead, if Slippery Pete gets into college 1, then he necessarily gets into college 2 (i.e. should he apply there).

**Solution:**

1. Let \( MB_{ij} \) = marginal benefit of adding college \( i \) to \( j \). Then by the given assumptions:

\[
MB_{21} = (1 - \alpha_1)\alpha_2 u_2 \\
MB_{12} = [\alpha_1 u_1 + (1 - \alpha_1)\alpha_2 u_2] - \alpha_2 u_2 = \alpha_1 (u_1 - \alpha_2 u_2)
\]

The solution is therefore:

- \( \alpha_1 u_1 \geq \max(c, \alpha_2 u_2) \) & \( MB_{21} < c \Rightarrow \{1\} \)
- \( \alpha_1 u_1 \geq \max(c, \alpha_2 u_2) \) & \( MB_{21} \geq c \) or \( \alpha_2 u_2 \geq \max(c, \alpha_1 u_1) \) & \( MB_{12} \geq c \Rightarrow \{1, 2\} \)
- \( \alpha_2 u_2 \geq \max(c, \alpha_1 u_1) \) & \( MB_{12} < c \Rightarrow \{2\} \)
- \( \alpha_1 u_1 < c \) & \( \alpha_2 u_2 < c \Rightarrow \emptyset \).

The application region boundaries are \( MB_{12} = c \) and \( MB_{21} = c \), as in the picture.

2. Let \( MB_{ij} \) = marginal benefit of adding college \( i \) to \( j \). Then by the given assumptions:

\[
MB_{21} = (\alpha_2 - \alpha_1) u_2 \\
MB_{12} = \alpha_1 u_1 + (\alpha_2 - \alpha_1) u_2 - \alpha_2 u_2 = \alpha_1 (u_1 - u_2)
\]

The solution continues as before. The application plots in \((\alpha_1, \alpha_2)\) space essentially reflect the fact that one applies nowhere if both \( \alpha_1 \) and \( \alpha_2 \) are small, one applies to college 2 if \( \alpha_2 \) is relatively high, to college 1 if \( \alpha_1 \) is relatively high, and to both if both are is high.
Figure 1: **Optimal College Applications.** The top and bottom panels depict the independent and perfectly correlated cases, respectively, and applies for any $1 \geq \alpha_1, \alpha_2 \geq 0$. The top left panel depicts (i) a dashed box, inside which applying anywhere is dominated; (ii) the indifference line for solo applications to colleges 1 and 2; and (iii) the marginal benefit curves $MB_{12} = c$ and $MB_{21} = c$ for adding colleges 1 or 2. The right panel shows the resulting optimal application regions. A student in the blank region $\Phi$ does not apply to college. He applies to college 2 only in the vertical shaded region $C_2$; to both colleges in the hashed region $B$, and to college 1 only in the horizontal shaded region $C_1$. (Just for fun, I include a diagonal line dividing what are stretch and safety schools. See if you can figure out why this is a good label.) In the bottom perfectly correlated case, the shaded regions depict the optimal portfolio choices for students when colleges observe perfectly correlated signals. The $MB_{12} = c$ curve separating regions $B$ and $C_2$ is now vertical. (Ignore the blue line; it is the “acceptance locus” in the source paper “Student Portfolios and the College Admissions Problem” by Greg Lewis, Hector Chade and Lones Smith, in RESStud (2014).)
Part II

The extensive form game $\Gamma$ corresponds to the following story: First, player 1 chooses whether to be In or Out. Player 2, after observing player 1’s choice, decides whether to be In or Out herself. For both players, the payoff to choosing Out is always 0, and the payoff to choosing In is 1 if the opponent is Out and is $-2$ if the opponent is In.

1. Draw $\Gamma$ and find all of its subgame perfect equilibria.

2. Present the reduced normal form $G(\Gamma)$ of $\Gamma$, and find all of its perfect equilibria and proper equilibria.

The extensive form game $\Gamma'$ corresponds to the following story: There are two players, 1 and 2, with player 1 moving first. At each decision node, the acting player chooses between Stop, which does not affect the acting player’s payoff but costs the opponent 3, and Continue, which benefits both players 1. The players alternate moves until either (I) a player chooses Stop or (II) Continue has been chosen 4 times, whichever comes first.

3. Draw $\Gamma'$ and find all of its subgame perfect equilibria.

4. Present the reduced normal form $G(\Gamma')$ of $\Gamma'$, and find all of its perfect equilibria and proper equilibria.

5. Taking advantage of your answers to the previous parts, describe (a) the extent to which perfect equilibrium captures backward induction in perfect information games, and (b) the extent to which proper equilibrium captures backward induction in perfect information games. In addition, provide an intuitive explanation of how the differences between the definitions of perfect and proper equilibrium lead to the differences in your answers to (a) and (b).

Solution sketch:

1. 

2. 

The unique subgame perfect equilibrium of $\Gamma$ is $(I, oi)$. 

2.
Since all strategies other than $oi$ are weakly dominated for player 2, they cannot be used in a perfect equilibrium. Thus $(I,oi)$ is the only perfect equilibrium of $G(\Gamma)$. This is also the only proper equilibrium of $G(\Gamma)$, since every proper equilibrium is perfect (or since all proper equilibria of $G(\Gamma)$ correspond to subgame perfect equilibria of $\Gamma$).

The unique subgame perfect equilibrium of $\Gamma'$ is $((C,C), (c,c))$.

Since this is a two-player game, a perfect equilibrium is a Nash equilibrium that does not use weakly dominated strategies. The only weakly dominated strategy is $cs$ for player 2. Once this is removed, computing the best response correspondences and matching
best responses shows that Nash equilibria of the game that remains are \((CC, cc)\) and \((S, \alpha s + (1 - \alpha)cc)\) with \(\alpha \geq \frac{2}{3}\). Since all proper equilibria of \(G(\Gamma')\) correspond to subgame perfect equilibria of \(\Gamma'\), the unique proper equilibrium of \(G(\Gamma')\) is \((CC, cc)\).

5. Perfect equilibrium captures subgame perfection in perfect information games with the single-move property, like \(\Gamma\), but unlike \(\Gamma'\). Proper equilibrium captures subgame perfection in all perfect information games (since all proper equilibria correspond to sequential equilibria in general extensive form games). Perfect equilibrium requires robustness against some completely arbitrary perturbations, while proper equilibrium requires robustness against perturbations that put lower orders of magnitude on strategies with lower payoffs under the specified perturbations. Only the latter captures the full force of backward induction, since backward induction requires that a player make correct decisions not only at their initial decision nodes on a play path, but also at later nodes that their initial decision precludes.

For instance, in game \(\Gamma'\), subgame perfection requires player 1 to respond optimally at his second decision node even if he chooses \(S\) at his initial decision node; this determines player 2’s optimal choice at his first decision node. In \(G(\Gamma')\), perfect equilibrium does not say anything about the relative probabilities assigned to \(CS\) and \(CC\), but proper equilibrium does: if player 2 does not play \(cs\), then \(CC\) should be far more likely than \(CS\).
Part III

1. Mr. A and B each have preferences over goods X and Y: A has utility $\exp(XY)$ and B has utility $XY^2 + \log(XY^2)$. Firm $X$ has production technology $X = \sqrt{L}$ and firm $Y$ has production technology $Y = 2\sqrt{L}$, where $L$ is the labor employed by that firm. Mr. A is endowed with labor $L = 2$ and B with labor $L = 3$. Suppose also that consumer A owns a share $\theta_X \in (0, 1)$ and $\theta_Y \in (0, 1)$ of firm $X$'s and $Y$'s profits, respectively. The other shares are owned by B.

(a) Find the range of feasible production possibilities in $(X,Y)$-space.

(b) Assume that labor is numeraire. Find the market demand for $X$ and $Y$.

(c) Fully characterize the competitive equilibrium.

Solution:

(a) Notice that $L_X + L_Y = 5$ which implies that the PPF is:

$$X^2 + \frac{Y^2}{4} = 5$$

(b) Define incomes for A and B,

$$I^A = 2 + \theta_X \pi_X + \theta_Y \pi_Y, \quad I^B = 3 + (1 - \theta_X) \pi_X + (1 - \theta_Y) \pi_Y$$

Thus, consumer $i \in \{A, B\}$ solves $\max_{X,Y} U^i(X, Y)$ subject to $P_i X + P_Y Y = I^i$. Solving using the standard Cobb-Douglas formulas, we derive demands:

$$X_A(P_X) = \frac{I^A}{2P_X}, \quad Y_A(P_Y) = \frac{I^A}{2P_Y}, \quad X_B(P_X) = \frac{I^A}{2P_X}, \quad Y_B(P_Y) = \frac{2I^B}{3P_Y}$$

Hence the aggregate demand for X is:

$$X^D(P_X) = X_A(P_X) + X_B(P_X) = \frac{I^A}{2P_X} + \frac{I^A}{2P_X}$$

Likewise, the aggregate demand for Y is:

$$Y^D(P_Y) = Y_A(P_Y) + Y_B(P_Y) = \frac{I^A}{2P_Y} + \frac{2I^B}{3P_Y}$$

(c) Firm $X$ solves $\pi_X = \max_{L_X} P_X \sqrt{L_X} - L_X$, while firm $Y$ likewise solves $\pi_Y = \max_{L_Y} P_Y 2\sqrt{L_Y} - L_Y$. The FOCs yield the demand for labor of each firm, and also the supply of $X$ and $Y$ respectively:

$$L_X = \frac{P_X^2}{4}, \quad L_Y = P_Y^2 \implies X^S(P_X) = \sqrt{L_X} = \frac{P_X}{2}, \quad Y^S(P_Y) = 2\sqrt{L_Y} = 2P_Y$$
Consequently, firm $X$’s profits are $\pi_X = P_X^2/4$, and firm $Y$’s profits are $\pi_Y = P_Y^2$. Next, we solve for $P_X$ and $P_Y$ using the market clearing equations for $X$ and $Y$:

\[
X^D(P_X) = X_A(P_X) + X_B(P_X) = X^S(P_X) \iff \frac{I^A}{2P_X} + \frac{I^B}{3P_X} = \frac{P_X}{2}
\]

\[
Y^D(P_Y) = Y_A(P_Y) + Y_B(P_Y) = Y^S(P_Y) \iff \frac{I^A}{2P_Y} + \frac{2I^B}{3P_Y} = 2P_Y
\]

After some algebra we obtain:

\[
P_X = 2\sqrt{\frac{22 + 5\theta_Y}{12 + \theta_Y - \theta_X}}, \quad P_Y = \sqrt{\frac{38 - 5\theta_X}{12 + \theta_Y - \theta_X}}
\]

thereby, we can obtain the equilibrium demands $X_A, Y_A, X_B, Y_B, L_X, L_Y$; and the equilibrium supplies $X^S, Y^S$. 
Part IV

Consider the following interaction between an informed seller, two uninformed buyers, and one intermediary. The seller owns an object that has value \( v \) to the buyers. The seller and the intermediary have no value for the object. The seller knows \( v \), and the buyers do not. The intermediary initially does not know \( v \), but can learn \( v \) by running a costless test during the interaction if asked to do so by the seller (see below). Play proceeds as follows:

1. The intermediary sets a fee \( \phi \geq 0 \) and commits to a disclosure rule \( \rho: [0,1] \rightarrow [0,1] \), which for each possible value \( w \) of the object specifies a report \( \rho(w) \).
2. Nature draws the actual value \( v \) from a uniform \((0,1)\) distribution.
3. Having observed \( \phi, \rho, \) and \( v \), the seller decides whether to hire the intermediary, i.e., whether to pay the fee and have the object tested; if the seller does so, the intermediary tests the object, observes \( v \), and publicly announces \( r = \rho(v) \).
4. Buyers observe \( \phi, \rho, \) whether the intermediary was hired, and, if so, the intermediary’s report \( r \). They then simultaneously make price offers \( p_1, p_2 \in [0,1] \) for the object.
5. The seller chooses the buyer to whom he will sell the object.

The seller and the intermediary each maximize expected dollar income. Each buyer maximizes her expected utility, where her utility is \( v - p \) for obtaining an object of value \( v \) at price \( p \), and is 0 for not obtaining the object and paying nothing.

(i) In this game, what constitutes (a) a pure strategy for the intermediary, (b) a pure strategy for the seller, and (c) a pure strategy for buyer \( i \in \{1,2\} \)?

(ii) Suppose we do not allow the intermediary to participate in the interaction, so that stages (1) and (3) are eliminated from the game. Describe the pure-strategy subgame perfect equilibria of this restricted game.

(iii) Suppose we let the intermediary participate, but require it to use a fully revealing disclosure rule: \( \rho(v) = v \) for all \( v \in [0,1] \). Under this restriction, find all pure-strategy weak sequential equilibria in which there is a threshold type \( \hat{v} \) such that sellers whose type is above \( \hat{v} \) hire the intermediary, and sellers whose type is below \( \hat{v} \) do not.

(iv) Return to the original model in which the intermediary chooses a disclosure rule. Construct a pure-strategy weak sequential equilibrium in which all types of seller hire the intermediary, and the intermediary chooses a completely uninformative disclosure rule: for instance, \( \rho(v) = 1 \) for all \( v \in [0,1] \). (Aside: It is possible to show that all weak sequential equilibria are of this form.)

(v) Compare the efficiency properties of the equilibrium outcomes in parts (ii), (iii), (iv). Also, find the intermediary’s expected equilibrium payoffs in parts (iii) and (iv), and compare each seller type’s equilibrium payoffs from parts (ii), (iii), (iv).
Solution: This question is based on Lizzeri (1999).

1. A pure strategy for the intermediary specifies a fee $\phi \geq 0$ and a disclosure rule $\rho : [0, 1] \to [0, 1]$.

A pure strategy for the seller is a pair of functions: a function from triples $(\phi, \rho, v)$ to decisions $d \in \{\text{yes, no}\}$, and a function from sextuples $(\phi, \rho, v, d, p_1, p_2)$ to choices of the buyer to whom to sell the object, $b \in \{1, 2\}$.

A pure strategy for buyer $i$ is a function from quadruples $(\phi, \rho, d, r)$, where $r$ equals “no report” if $d = \text{no}$ and $r$ equals the seller’s report otherwise, to prices $p_i \in [0, 1]$.

2. We work backward. In stage 5, the seller will sell the good to one of the sellers who makes the highest price offer.

In stage 4, both buyers will choose a price equal to the expected value of the object given their information. (If buyer $i$ offered a price higher than this, then buyer $j$ would make a lower offer, so buyer $i$ would obtain negative utility; if buyer $i$ offered a price lower than the expected value, she would be outbid by buyer $j$. If buyer $i$ offered this expected value and buyer $j$ something less, then buyer $i$ could profitably reduce her offer.) Since all seller types are willing to sell, it must be that $p_i = p_j = \int_0^1 w \, dw = \frac{1}{2}$. (For the buyers’ strategies to be optimal, the seller’s strategy must be such that each buyer earns expected utility 0; this is true, for instance, if the seller always sells to buyer 1.)

3. In stage 5, the seller will sell the good to one of the sellers who makes the highest price offer.

In stage 4, both buyers will choose a price equal to the expected value of the object given their information. Thus $p_i = p_j = r$ if a report $r \in [0, 1]$ is issued. If no report is issued, then $p_i = p_j$ is the expected value of the object conditional on this fact. Thus if the seller uses a threshold rule with threshold $\hat{v}$, then $p_i = p_j = (\int_0^{\hat{v}} w \, dw)/\hat{v} = \hat{v}/2$. (For a threshold rule with threshold $\hat{v}$ to be optimal for the seller, it must be that $v - \phi = \hat{v}/2$, and hence that $\phi = \hat{v}/2$.

In stage 3, a type $v$ seller obtains payoff $v - \phi$ for hiring the intermediary and payoff $\hat{v}/2$ for not doing so. Thus for a threshold rule with threshold $\hat{v}$ to be optimal for the seller, it must be that $v - \phi = \hat{v}/2$, and hence that $\phi = \hat{v}/2$.

In stage 1, an intermediary who chooses a fee of $\phi \in [0, \frac{1}{2}]$ will induce a threshold of $\hat{v} = 2\phi$, and so will be hired by fraction $1 - 2\phi$ of the sellers, and so obtain a profit of $\phi - 2\phi^2$. (Choosing $\phi > \frac{1}{2}$ will lead to never being hired.) Thus the intermediary optimizes by choosing $\phi = \frac{1}{4}$, inducing a threshold of $\hat{v} = \frac{1}{2}$.

4. If all seller types hire the intermediary, then the “no report” information set is unreached, and we must specify the buyers’ beliefs there. To make it easiest to support the equilibrium, we suppose that in this event, the buyers believe that the seller’s type is 0. Thus the buyers offer a price of 0 in this case. Similarly, we suppose that if a report of $r < 1$ is sent, the buyers believe that the seller’s type is 0, and so offer a price of 0.

If all seller types hire the intermediary, and the intermediary always issues a report of 1. Thus as in part (ii), the buyers offer $p_i = p_j = \frac{1}{2}$. Thus all seller types are strictly better off paying any fee $\phi < \frac{1}{2}$ to get certified rather than going uncertified and
receiving price offers of 0. They are indifferent when the fee is $\phi = \frac{1}{2}$; to obtain an equilibrium we require them to pay the fee in this case.

If the intermediary charges a fee of $\phi = \frac{1}{2}$ and chooses the fully revealing disclosure rule, it is hired by all seller types and so earns an expected payoff of $\frac{1}{2}$. Since $\frac{1}{2}$ is the total expected surplus that exists in the market, the intermediary could not obtain a higher expected payoff by choosing a different fee and disclosure rule if the seller and buyers make rational choices on the path of play.

In all three cases, the good is always sold, so equilibrium is always efficient. In case (ii) there is no intermediary, and sellers of all types get payoff $\frac{1}{2}$. In case (iii), the intermediary gets an expected payoff of $\frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}$; sellers of types $v \geq \frac{1}{2}$ get payoff $v - \frac{1}{4}$, and sellers of types below $\frac{1}{2}$ get payoff $\frac{1}{4}$ (thus the ex ante expected seller payoff is $\frac{1}{2} \cdot \frac{1}{4} + \int_{1/2}^{1} (w - \frac{1}{4}) \, dw = \frac{1}{8} + \frac{1}{4} = \frac{3}{8}$). In case (iv) the intermediary gets expected payoff $\frac{1}{2}$, and all seller types get nothing. Sellers of types $v > \frac{3}{4}$ like case (iii) better than case (ii); sellers of types $v < \frac{3}{4}$ like case (ii) better than case (iii), and all seller types prefer both of these cases to case (iv).