University of Wisconsin
Microeconomics Prelim Exam
Monday, August 7, 2012: 9AM - 2PM

Questions and Solutions

• There are four parts to the exam. All four parts have equal weight.

• Answer all questions. No questions are optional.

• Hand in 12 pages, written on only one side.

• Write your answers for different parts on different pages. So do not write your answers for questions in different parts on the same page.

• Please place a completed label on the top right corner of each page you hand in. On it, write your assigned number, and the part of the exam you are answering (I,II,III,IV). Do not write your name anywhere on your answer sheets!

• Show your work, briefly justifying your claims. Some solutions might be faster done by drawing a suitable diagram.

• You cannot use notes, books, calculators, electronic devices, or consultation with anyone else except the proctor.

• Please return any unused portions of yellow tablets and question sheets.

• There are five pages on this exam, including this one. Make sure you have all of them.

• Best wishes!
Part I

Suppose that a consumer’s preference ordering on $\mathbb{R}^3_+$ can be represented by the utility function
\[ u(x) = \alpha x_1 x_2 + \log(x_3) \]
where $\alpha > 0$.

The consumer has wealth $w$ and faces (positive) prices $p_1, p_2, p_3$ for the three goods.

1. Suppose $\alpha = 2$, and $p_1 = 2, p_2 = 2, p_3 = 20, w = 5$. What is the optimal consumption plan?

2. For any given value of $\alpha$, does there exist a budget constraint such that it is optimal to spend equal amounts on each of the three goods?

3. Is it possible that the consumer would choose to consume a smaller quantity of one of the goods when wealth increases?

Solution Heuristics

Quasilinear utility means that the utility function is additively separable, and all goods except one have strictly decreasing marginal utility, while this one good has constant marginal utility. Once this good enters the optimal consumption plan, it absorbs all income effects, in the sense that increases in wealth are used to buy more of this good, with no changes in the quantities of the other good. If the price of this good is high, or wealth is low, there is a corner solution in which the marginal utility per dollar is the same for all but this one good, and this good is not consumed, because the marginal utility per dollar is too low.

In the problem here, something similar happens, but the analysis is a bit more complicated.

The marginal utility per dollar spent on goods 1 and 2 is increasing, starting from zero (if the money is equally divided between these two goods). Once these goods enter the optimal consumption plan, they absorb all income effects and then some: increases in wealth actually increase the marginal utility per dollar spent on these goods, with the result that the quantity of the other good decreases so as to raise marginal utility per dollar spend on that good to the new level. Thus good 3 is inferior (at higher wealth levels).

Buying only good 3 is always locally optimal, because marginal utility per dollar is positive, and it is zero for goods 1 and 2 at the corner.
Analysis

Since the marginal utility of good 3 is infinite at zero, $x_3$ must be positive at an optimum. Also, $x_1 > 0$ implies $x_2 > 0$, and vice versa, and at an interior solution, marginal utilities per dollar must be the same for all three goods. For the first two goods, this implies

$$\frac{x_1}{p_2} = \frac{x_2}{p_1}$$

so expenditure on these two good must be equal (and this is also true at a corner solution).

Let $y = w - p_3x_3$ be the amount spent on goods 1 and 2. Then utility is

$$u = \frac{\alpha y^2}{4p_1p_2} + \log \left( \frac{w-y}{p_3} \right)$$

$$= \frac{1}{2} \frac{y^2}{Q} + \log (w-y) - \log (p_3)$$

where $Q = 2^{\frac{p_1}{p_2}} > 0$.

Let

$$\phi (y) = \frac{1}{2} \frac{y^2}{Q} + \log (w-y)$$

The optimal solution is found by maximizing this with respect to $y$, and then setting

$$x_1 = \frac{y^*}{2p_1}$$

$$x_2 = \frac{y^*}{2p_2}$$

$$x_3 = \frac{w-y^*}{p_3}$$

The function $\phi$ has derivatives

$$\phi' (y) = \frac{y}{Q} - \frac{1}{w-y}$$

$$\phi'' (y) = \frac{1}{Q} - \frac{1}{(w-y)^2}$$

At an interior maximum the FOC can be written as

$$y (w-y) = Q$$
and \( \phi''(y) \leq 0 \) so
\[
(w - y)^2 \leq Q
\]

Taken together, these imply
\[
(w - y)^2 \leq y(w - y)
\]
and since \( y < w \) this reduces to \( w - y \leq y \), so at an interior optimum,
\[
y^* \geq \frac{w}{2}
\]
so the function \( \phi(y) \) has at most two local maxima, including one at 0. Since \( y(w - y) \leq \frac{w^2}{4} \), there is no interior maximum if \( Q \geq \frac{w^2}{4} \). So the corner solution is optimal if wealth is low or \( Q \) is large (i.e. if the first two goods are expensive, or if they have low weight in the utility function).

If \( z = w - y = p_3 x_3 \), the interior maximum is where the rectangular hyperbola \( yz = Q \) intersects the budget line \( y + z = w \); if these don’t intersect there is no interior maximum (so the corner solution is optimal), and if they do intersect, there are two (symmetric) intersections, and the relevant one has \( y \geq z \). An increase in income moves the budget line out without affecting the hyperbola, so the intersection moves down along the hyperbola to the right, meaning that \( y \) increases and \( z \) decreases.

**Answers**

1. Suppose \( \alpha = 2 \), and \( p_1 = 2, p_2 = 2, p_3 = 20, w = 5 \). What is the optimal consumption plan?
   At any interior optimum,
   \[
   \phi(y) = \frac{y^2}{2Q} + \log(w - y)
   \]
   \[
   = \frac{1}{2} \frac{y}{w - y} + \log(w - y)
   \]
   Here \( Q = 4 \), so the FOC is solved by setting \( y = 4 \), with \( \phi(4) = 2 \), and \( \phi(0) = \log(5) \), so the interior solution is optimal (\( 2 > \log(5) \) because \( e^2 > (1 + \frac{1}{2})^2 > 5 \)).

2. For any given value of \( \alpha \), does there exist a budget constraint such that it is optimal to spend equal amounts on each of the three goods?
No. This would imply

\[ Q = y(w - y) = \frac{2w^2}{9} \]

Then

\[ \phi(y) = \frac{1}{2} \frac{4w^2}{3w^2} + \log(w) - \log(3) = 1 + \phi(0) - \log(3) \]

So setting \( y = 0 \) gives higher utility (since \( \log(3) > 1 \)).

3. Is it possible that the consumer would choose to consume a smaller quantity of one of the goods when wealth increases?

Yes. For example, in the answer to the first part, \( x_3 = \frac{1}{20} \). But if \( w \) is reduced from 5 to 3, then \( Q > \frac{w^2}{4} \), so the corner solution is optimal, meaning that \( x_3 = \frac{3}{20} \).
Part II

Two customer service workers each choose effort levels \( e_i \in [0, 10] \) to exert in helping a customer. The cost of effort is \( c_1(e_1) = (e_1)^2/15 \) for worker 1 and \( c_2(e_2) = (e_2)^2/10 \) for worker 2. The customer’s satisfaction level equals the total effort that the workers exert, up to a maximum satisfaction level of 10. Each worker’s payoff is the difference between the customer’s satisfaction level and his own effort cost.

Consider the game \( G \) in which the workers simultaneously choose their effort.

1. Describe each worker’s payoff function in \( G \).
2. Describe each player’s best response correspondence in \( G \).
3. Compute all pure Nash equilibria of \( G \).

Now we consider sequential-move versions of the interaction. In game \( \Gamma_2 \), first player 1 selects an effort level. Then player 2, after observing player 1’s choice, chooses her own effort level.

4. Completely describe all pure strategy subgame perfect equilibria of \( \Gamma_2 \).

In \( \Gamma_3 \), the sequence of events starts with those in \( \Gamma_2 \). But after player 2 moves, player 1 observes her choice and decides whether to exert some additional effort, \( a_1 \in [0, 10] \). His cost of effort in \( \Gamma_3 \) is given by \( c_1(t_1) = (t_1)^2/15 \), where \( t_1 = e_1 + a_1 \) is the sum of his initial effort choice and his additional effort choice.

5. What is a pure strategy for player 1 in \( \Gamma_3 \)? What is a pure strategy for player 2? (You are free to state your answers in English rather than notation, but either way, your answers should be explicit.)
6. Completely describe all pure strategy subgame perfect equilibria of \( \Gamma_3 \).

Solution

1. The players’ payoff functions are

\[
\begin{align*}
u_1(e_1, e_2) &= \begin{cases} 
    e_1 + e_2 - (e_1)^2/15 & \text{if } e_1 + e_2 \leq 10, \\
    10 - (e_1)^2/15 & \text{if } e_1 + e_2 > 10, 
\end{cases} \\
u_2(e_1, e_2) &= \begin{cases} 
    e_1 + e_2 - (e_2)^2/10 & \text{if } e_1 + e_2 \leq 10, \\
    10 - (e_2)^2/10 & \text{if } e_1 + e_2 > 10, 
\end{cases}
\end{align*}
\]
2. For fixed \( e_2 \), the first case of \( u_1(e_1, e_2) \) above is maximized when \( 1 - 2e_1/15 = 0 \), or equivalently when \( e_1 = 7\frac{1}{2} \). But if \( e_2 > 2\frac{1}{2} \), player 1 would prefer to choose \( e_1 = 10 - e_2 \), which generates the same benefit at a lower cost. Thus

\[
b_1(e_2) = \begin{cases} 
7\frac{1}{2} & \text{if } e_2 \leq 2\frac{1}{2}, \\
10 - e_2 & \text{if } e_2 > 2\frac{1}{2}.
\end{cases}
\]  

(1)

Similarly,

\[
b_2(e_1) = \begin{cases} 
5 & \text{if } e_1 \leq 5, \\
10 - e_1 & \text{if } e_1 > 5.
\end{cases}
\]  

(2)

3. Draw the graphs of the two best response correspondences. The Nash equilibria are their intersection, namely, all pairs \((x, 10 - x)\) with \( x \in [5, 7\frac{1}{2}] \).

4. We use backward induction. When player 2 moves, she will play a best response to player 1’s initial action as described by (2). So player 1 seeks to maximize

\[
u_1(e_1, b_2(e_1)) = \begin{cases} 
e_1 + 5 - (e_1)^2/15 & \text{if } e_1 \leq 5, \\
10 - (e_1)^2/15 & \text{if } e_1 > 5,
\end{cases}
\]

This is maximized when \( e_1 = 5 \). Thus the unique subgame perfect equilibrium is \( s_1 = 5, s_2(e_1) = b_2(e_1) \), where the function \( b_2 \) is defined in (2).

5. A pure strategy for player 1 specifies his initial effort level, as well as his additional effort level as a function of both his initial effort level and player 2’s effort level. A pure strategy for player 2 specifies her effort level as a function of player 1’s initial effort level.

6. We use backward induction. At the end of the game, as in (1), it is optimal for player 1 to target the sum of his efforts \( t_1 = e_1 + a_1 = 7\frac{1}{2} \) if \( e_2 \leq 2\frac{1}{2} \), and to target the total effort \( e_1 + e_2 + a_1 = 10 \) if \( e_2 > 2\frac{1}{2} \). But if player 1’s initial effort exceeds these requirements, he should make no additional effort. All told, player 1’s strategy at the end of the game in a subgame perfect equilibrium is

\[
\beta_1(e_1, e_2) = \begin{cases} 
7\frac{1}{2} - e_1 & \text{if } e_2 \leq 2\frac{1}{2} \text{ and } e_1 \leq 7\frac{1}{2}, \\
10 - e_1 - e_2 & \text{if } e_2 > 2\frac{1}{2} \text{ and } e_1 \leq 10 - e_2, \\
0 & \text{if } e_2 \leq 2\frac{1}{2} \text{ and } e_1 > 7\frac{1}{2}, \text{or } e_2 > 2\frac{1}{2} \text{ and } e_1 > 10 - e_2.
\end{cases}
\]

In the middle of the game, player 2 realizes that if she chooses \( e_2 \geq 2\frac{1}{2} \), player 1’s additional effort will bring the total effort up to at least 10. Thus player 2
should never choose an effort level higher than $2\frac{1}{2}$. If player 1 chose an effort $e_1 \leq 7\frac{1}{2}$ in period 1, then this is exactly what player 2 should do. But if player 1 chose a higher effort level, then player 2 need only exert enough effort to bring the total effort up to 10. Thus, player 2’s strategy in any subgame perfect equilibrium is described by

$$
\beta_2(e_1) = \begin{cases} 
2\frac{1}{2} & \text{if } e_1 \leq 7\frac{1}{2}, \\
10 - e_1 & \text{if } e_1 > 7\frac{1}{2}.
\end{cases}
$$

In the initial period, player 1 should never choose an effort higher than $7\frac{1}{2}$, since choosing $7\frac{1}{2}$ is enough to guarantee that the total effort will be 10. But any $e_1 \in [0, 7\frac{1}{2}]$ will cause player 2 to choose $e_2 = 2\frac{1}{2}$, which player 1 will follow by choosing $a_1 = 7\frac{1}{2} - e_1$. Thus any $e_1 \in [0, 7\frac{1}{2}]$ can be chosen initially in subgame perfect equilibrium.
Part III

1. You are traveling to Ottawa and stupidly you left the hotel choice until the last minute. Now you need to find a hotel to stay at for one night. You pay for a new travel service called www.fixedstigler.com. This service lets you pay a flat fee of $100 for any hotel. If you spend $nc$ dollars, you get a listing of $n$ such hotels. Hotels vary in amenities, so that the value of any hotel is $100 + v$, where $v$ is uniformly distributed on $[0, 100]$, independently across hotels. So the chance that any hotel yields you a net value or surplus of at least $v$ is $1 - v/100$.

Assume that your goal in choosing $n$ is to maximize the surplus, net of costs. What is the demand curve $n(c)$? What is a simple approximation for $n(c)$?

2. A total of 4000 cars wish to travel from start to end. Assume first that no road joins A to B. Then there are two routes from start to end. The travel time in minutes on the road from Start to A is the number of travelers ($T$) divided by 100, and on the road from Start to B is a constant 45 minutes. Then these road times switch from A and B to end, as indicated.

(a) What is the competitive equilibrium driving time from start to end?

(b) Now assume that a two-way road joining A and B is built that takes 10 minutes. (Ignore the arrows on that road.) What is the competitive equilibrium driving time? Comment.

(c) What driving choices would a social planner enforce who cared about total drive times of all drivers? How exactly would he do it?
Solution

1. One will choose the highest net value $v$. Now, the cdf of the maximum of these net values is the chance $F(v) = (v/100)^n$ that all $n$ searches yield a net value below $v$. Thus, the expected maximum is the area over the survivor:

$$\int_0^{100} 1 - F(v) dv = \int_0^{100} 1 - (v/100)^n dv = 100 - \frac{100^{n+1}}{(n+1)100^n} = 100 - \frac{100}{n+1}$$

The marginal benefit is the differenced form of this net benefit. Since the marginal cost is $c$, the FOC is discrete. One chooses $n$ if

$$\frac{100}{n(c)[n(c) - 1]} \geq c > \frac{100}{n(c)[n(c) + 1]} \quad \Rightarrow \quad n(c) \approx \frac{10}{\sqrt{c}}$$

This implicit equation is the simplest precise formula we have, apart from using the quadratic formula.

2. (a) In equilibrium, travel times will equalize. Assume that $m$ take the A route, and $n$ take the B route. Then we find $m + n = 4000$ and

$$\frac{m}{100} + 45 = \frac{n}{100} + 45$$

Thus, $m = n = 2000$, and the total time is 65 minutes.

(b) At the above equilibrium, when arriving at point A, a driver is tempted to switch, because $45 > 10 + 2000/100$. In fact, let the numbers of drivers on the A route be initially $m$ and then $M$, and on the B route, initially $n$ and then $N$.

$$45 = 10 + \frac{N}{100} \quad \Rightarrow \quad N = 3500$$

So $M = 500$. Likewise, $m = 3500$ and $n = 500$. The total time is now $45 + 35 = 80$. A new option has hurt drivers! This is Braess’s paradox.

(c) The social planner would choose $m, M$ to minimize

$$L = m(m/100) + 45(4000 - m) + (m - M)10 + 45M + (4000 - M)^2/100 + \lambda(m - M)$$

$\Rightarrow$ FOC’s:

$$\frac{m}{50} - 45 + 10 - \lambda = 0 = (M - 4000)/50 + 45 - 10 + \lambda$$

Thus, $m = 1750 - 50\lambda$ and $M = 2250 + 50\lambda$ and if $m \geq M$, as $\lambda \geq 0$ by the saddle point property, given the complementary slackness condition $\lambda(m - M) = 0$. The only solution is $m = M$, and $\lambda = -5$. That is, the planner should employ the road tax 5, measured in minutes. (The negative sign reflects the fact that the tax only need applies going one direction.) The planner could just impose a mandatory five minute wait before traversing the AB road.
Part IV

A firm can earn profits from two different activities undertaken by the worker. The firm’s return $\pi_1$ from activity 1 is a deterministic function of the worker’s effort on this activity. If the worker exerts high effort $e_h$ on activity 1, then $\pi_1 = Y$; if he exerts low effort $e_l$, then $\pi_1 = Z$, where $Z < Y$. The firm’s return $\pi_2$ from activity 2 is the following stochastic function of the worker’s effort: If the worker exerts high effort $e_h$ in activity 2, then $\pi_2 = X$ and $\pi_2 = 0$ with equal chances 0.5. If the worker puts in low effort $e_l$, then $\pi_2 = 0$ always.

The worker can exert high effort on at most one activity. Only three effort combinations are possible: effort $e_l$ in both activities, effort $e_h$ in activity 1 and $e_l$ in activity 2, or effort $e_h$ in activity 2 and $e_l$ in activity 1. Exerting high effort is costly for the worker. Specifically, the worker’s utility as a function of the contractually specified wage $w$ is $\sqrt{w}$ if he puts low effort into both activities and $\sqrt{w} - g$ if he exerts high effort on one activity, where $g$ is the utility effort cost. The worker’s utility if he does not work for the firm is 0. Assume that $0.5X > Y - Z > g^2$. Thus, the expected marginal return of high effort in activity 2 exceeds the marginal return of high effort in activity 1. The firm is risk neutral.

1. Assume that the firm observes the worker’s effort level in each activity. Characterize the contract (wages and effort levels) that the firm will offer.

2. Suppose that the firm observes both returns $\pi_1$ and $\pi_2$ but not the worker’s effort choice. Find the profit-maximizing contracts for inducing each combination of efforts in both activities. Under what conditions on $X$, $Y$ and $Z$ is the effort combination chosen by the firm different from that in part (a)?

3. For each combination of efforts in parts (a) and (b), provide economic intuition why the wage schedules are the same/different. Verify whether the monotone likelihood ratio property holds and use this in your explanation. Interpret the risk sharing properties of the optimal contracts.

Solution

(1) The first-best contract: The firm’s optimization involves choosing a combination of efforts in both activities that maximizes expected profit subject to the participation/individual rationality (IR) constraint for the worker,

$$\sqrt{w} - g \geq 0 \quad (IR).$$
To induce low effort $e_l$ in both activities, the firm will pay the worker 0. To induce high effort $e_h$ in one of the activities, the firm will pay the worker $g^2$ so that the participation constraint is binding. To determine in which activity, if any, the firm will induce $e_h$, let’s compare the firm’s payoffs: The firm’s profit from $e_h$ in activity 1 is $Y - g^2$ and its profit from $e_h$ in activity 2 is $0.5X + Z - g^2$. The profit from low effort in both activities is $Z$. So by the assumptions, the profit maximizing choice for the firm is high effort in activity 2.

(2) The second-best contract: If the firm chooses to induce the worker to choose low effort $e_l$ in both activities, the profit maximizing wage is constant across return realizations and, to ensure that the IR binds, the constant wage is 0. In this case, the firm’s profit is $Z$.

If the firm chooses to induce the worker to choose high effort $e_h$ in activity 1, since effort in activity is effectively observable, the firm can induce $e_h$ by offering a wage contract in which the worker is paid $w = g^2$ if $\pi_2 = Y$ and $w = 0$ otherwise. This yields profit equal to $Y - g^2$. Given that $Y - Z > g^2$, this combination of efforts is preferred by the firm to low effort in both activities.

If the firm chooses to induce the worker to choose high effort $e_h$ in activity 2, the firm’s optimization now involves choosing the wages to maximize expected profit subject to the individual rationality constraint (IR) and incentive constraint (IC) for the worker. Let $w_X$ and $w_0$ be the wage paid to the worker, conditional on the firm observing returns $\pi_2 = X$ and $\pi_2 = 0$, respectively. The participation constraint is

$$0.5\sqrt{w_X} + 0.5\sqrt{w_0} - g \geq 0 \ (IR)$$

and the incentive constraint is

$$0.5\sqrt{w_X} + 0.5\sqrt{w_0} - g \geq \sqrt{w_0} \ (IC).$$

Since both constraints bind at the optimum,

$$0.5\sqrt{w_X} + 0.5\sqrt{w_0} - g = 0,$$

$$0.5\sqrt{w_X} + 0.5\sqrt{w_0} - g = \sqrt{w_0},$$

this gives $w_0 = 0$. Substituting $w_0$ into the IR constraint, we have $0.5\sqrt{w_X} = g$, or $w_X = 4g^2$. Hence, the firm’s expected profit from this combination of effort levels is $0.5(X - 4g^2) + Z$.

The firm will offer a contract that induces a different combination of effort levels from the first-best if

$$0.5(X - 4g^2) + Z > Y - g^2,$$
or $0.5X - Y + Z > g^2$.

(3) In the first-best, the optimal contract then equalizes the ratios of marginal utilities of the firm and the worker across states (return realizations). Given the risk neutrality of the firm and the risk aversion of the worker, the firm fully insures the worker against the risk by offering a constant wage schedule. The desired combination of effort levels can be induced with a wage schedule conditional on effort levels directly and an incentive constraint does not enter the firm’s optimization.

When effort is not observable, the wage schedule cannot condition on it directly. Hence, if the firm chooses to induce high effort in activity 2, in which return is a stochastic function of effort, both participation and incentive constraints affect the wage contract, which is no longer constant across states. The monotone likelihood ratio property holds (in activity 2, the likelihood ratio of high-to-low output is higher conditional on high effort than on low effort, $0.5 > 0.1$) and dictates that the worker should be paid more in higher-return state. Thus, the profit maximizing contract that induces high effort in activity 2 uses the correlation between effort and output. The worker now bears risk, as a result of a trade-off between risk sharing and incentives: optimal risk sharing recommends that the wage does not vary too much across states, whereas incentive provision recommends that the wage does depend on the state. If the firm chooses to induce high effort in activity 1, in which return is a deterministic function of effort, there is no risk to share and the incentives are taken care of by participation constraint.