University of Wisconsin
Microeconomics Prelim Exam with Solution Sketches
Friday, August 2, 2013: 9AM - 2PM

• There are four parts to the exam. All four parts have equal weight.
• Answer all questions. No questions are optional.
• Hand in 12 pages, written on only one side.
• Write your answers for different parts on different pages. So do not write your answers for questions in different parts on the same page.
• Please place a completed label on the top right corner of each page you hand in. On it, write your assigned number, and the part of the exam you are answering (I,II,III,IV). Do not write your name anywhere on your answer sheets!
• Show your work, briefly justifying your claims. Some solutions might be faster done by drawing a suitable diagram.
• You cannot use notes, books, calculators, electronic devices, or consultation with anyone else except the proctor.
• Please return any unused portions of yellow tablets and question sheets.
• There are five pages on this exam, including this one. Make sure you have all of them.
• Best wishes!
Part I

Suppose that before going on a stellar quiz show, Lones might study names of stars. Each star’s name he learns costs him \( c > 0 \). The quiz show will ask him a sequence of questions, and he must supply at least \( k \) right answers to win the prize \( \pi > c \); otherwise, the prize is 0. Each question he studies will be tested with chance \( q \in (0, 1) \), independently across questions. Assume that Lones optimal learns the names of \( n \) stars to maximize his net payoff.

(a) Assume \( k = 1 \). What is gross value of studying the \( n \) questions? How many questions will Lones study? Provide a characterizing inequality and an approximation.

(b) How does the demand \( n \) by Lones change in the chance \( q \)?

(c) Assume \( k = 2 \). What is value of studying the \( n \) questions? What is the range of possible \( q \) such that Lones will never choose to study \( n = 2 \) questions?

Hint: What does the plot of the marginal value of studying an extra question look like? To answer this, you will need to show an inequality.

Solution sketch:

(a) To get at least one correct answer, one must not get zero answers. To wit, the value is

\[ V(n) = \pi[1 - (1 - q)^n], \]

by independence. So the marginal value is

\[ \Delta V(n) = V(n) - V(n - 1) = \pi[1 - (1 - q)^{n-1} - (1 - q)^n] = \pi(1 - q)^{n-1}q \]  

This strictly falls in \( n \). Thus, demand \( n^* \) occurs when \( (1 - q)^{n^*-1}q > c > (1 - q)^{n^*-2}q \). Taking logs, \( (1 - q)^{n^*-1}q \approx c \) or \( n^* \approx 1 + \log[c/(q\pi)]/(1 - q) \).

(b) As \( q \) rises, demand first rises and then falls. For we can sign the derivative in \( q \) because \( (1 - q)(-1/q) + \log[c/(q\pi)] = 1 - 1/q + \log[c/(q\pi)] \) turns negative as \( q \uparrow 1 \), for \( c < \pi \).

(c) To get at least two correct answers, one must not get zero or one answer. Thus, the value is

\[ V(n) = \pi[1 - (1 - q)^n - nq(1 - q)^{n-1}], \]

by independence. Now,

\[ n(1 - q)^{n-2} - nq(1 - q)^{n-1} = q(1 - q)^{n-2}[(n - 1) - n(1 - q)] = q(1 - q)^{n-2}(nq - 1) \]

Adding together terms (1) and (2), the marginal value is

\[ \Delta V(n) = (1 - q)[(1 - q) + q(nq - 1)] = \pi(1 - q)^{n-2}[1 - q + q(nq - 1)] \]

For instance, \( \Delta V(2) = \pi(1 - 2q + 2q^2) \), and

\[ \Delta V(3) = \pi(1 - q)[(1 - q) + q(3q - 1)] = \pi(1 - q)(1 - 2q + 3q^2) \]

Thus, for large \( q < 1 \), the marginal value is initially strictly rising, since:

\[ \Delta V(3) - \Delta V(2) = \pi[(1 - q)(1 - 2q + 3q^2) - (1 - 2q + 2q^2)] = \pi q[5q - 1 - 3q^2] \]

The marginal value rises from 2 to 3 for \( q \) close to 1, and in fact whenever \( q \) exceed the root \( q = (5 - \sqrt{25 - 12})/6 = (5 - \sqrt{13})/6 \approx 0.232 \). On the other hand, the marginal value \( \Delta V(n) \) vanishes as \( n \to \infty \). Thus, \( n = 2 \) will never solve the optimization condition, since the second order condition fails.
Part II

The federal government is conducting a single second-price auction to sell a bundle consisting of two parcels of land, numbered 1 and 2, from two different oil-producing regions. The bidders are two firms, also numbered 1 and 2. Firm 1’s headquarters is close to parcel 1, and firm 2’s headquarters is close to parcel 2.

Before the auction, firm \( i \in \{1, 2\} \) sends a team of geologists to parcel \( i \) to determine its quality. Ex ante, the quality \( q_i \) of a parcel \( i \) follows a \( \text{uniform}(0, 1) \) distribution, and the quality levels of the two parcels are independent of one another. Firm \( i \) learns the quality of parcel \( i \), but firm \( j \) does not.

The value of parcel \( i \) to firm \( i \) is equal to the parcel’s quality. But because of the inconvenience of working at a remote location, the value of parcel \( j \) to firm \( i \) is its quality multiplied by \( \delta \in (0, 1) \). Firm \( i \)'s value for the bundle is the sum of the values it assigns to the parcels.

When the auction is run, each firm \( i \) places a bid \( b_i \in [0, 1+\delta] \) for the bundle of parcels. If there is a unique high bid, then the high bidder wins the bundle and pays a price equal to the bid of the low bidder. If the bids are equal, then the winner is determined by a coin toss and pays a price equal to the common bid.

(a) Define a Bayesian game that represents this auction.

Firm \( i \) is said to use a linear bidding strategy if its bid is a linear function of its quality observation: \( s_i(q_i) = \alpha_i q_i \) for some \( \alpha_i \geq 0 \).

(b) Compute the government’s expected revenue from the auction if the firms use linear bidding strategies.

(c) Show that choosing a linear bidding strategy with \( \alpha_i < 1 \) is a weakly dominated strategy for firm \( i \).

(d) Find the auction’s unique Nash equilibrium in undominated linear bidding strategies.

When firm \( i \) learns who won the auction, it also learns something about the signal that firm \( j \) received. A bidder who fails to account for this information in determining its optimal bid is sometimes said to suffer from the winner’s curse.

(e) If firm 2 follows the bidding strategy you found for it in part (iv), what would firm 1’s bidding strategy be if it suffered from the winner’s curse, but optimized correctly in other respects? Compare this strategy with the strategy you found for firm 1 in part (iv), and explain as precisely as possible why the strategies differ in the way that they do.

Solution:
(a) The Bayesian game is defined as follows:
\[
P = \{1, 2\} \\
A_i = [0, 1 + \delta] \\
T_i = [0, 1] \\
p \sim \text{uniform}([0, 1]^2) \\
u_i(b_i, b_j, q_i, q_j) = \begin{cases} \\
q_i + \delta q_j - b_j & \text{if } b_i > b_j, \\
\frac{1}{2}(q_i + \delta q_j - b_j) & \text{if } b_i = b_j, \\
0 & \text{if } b_i < b_j. \\
\end{cases}
\]

(b) The government’s expected revenue is the expected value of the lower bid. Clearly, if \(\alpha_1 = 0\) or \(\alpha_2 = 0\), then the lower bid will 0. Otherwise, let \(B_1\) and \(B_2\) be random variables representing the two bidders’ bids, and let \(L\) be a random variable representing the lower of the two bids. Then \(B_i \sim \text{uniform}(0, \alpha_i)\) and the bids are independent. If we assume without loss of generality that \(\alpha_1 \leq \alpha_2\), then \(L\) has support \([0, \alpha_1]\), and its c.d.f. is
\[
F_L(b) = P(L \leq b) = P(B_1 \leq b \text{ or } B_2 \leq b) = 1 - P(B_1 > b \text{ and } B_2 > b) = 1 - P(B_1 > b)P(B_2 > b) = 1 - \left(1 - \frac{b}{\alpha_1}\right)\left(1 - \frac{b}{\alpha_2}\right) = \left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2}\right)b - \frac{1}{\alpha_1\alpha_2}b^2
\]

Thus the expected value of \(L\), and hence the government’s expected revenue, is
\[
\int_0^{\alpha_1} b f_L(b) \, db = \int_0^{\alpha_1} b \left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2}\right)b - \frac{2}{\alpha_1\alpha_2}b^2 \, db = \left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2}\right)\frac{1}{2}(\alpha_1)^2 - \frac{2}{\alpha_1\alpha_2} \frac{1}{3}(\alpha_1)^3 = \frac{1}{2}\alpha_1 - \frac{1}{6}\alpha_2.
\]

(c) Choosing \(\alpha_i = a < 1\) is weakly dominated by choosing \(\alpha_i = 1\). To see this, fix \(q_i\). If player \(j\)’s bid is less than \(aq_i\), then player \(i\) wins whether he bids \(aq_i\) or \(q_i\). If player \(j\)’s bid is greater than \(q_i\), then player \(i\) loses whether he bids \(aq_i\) or \(q_i\). But if player \(j\)’s bid \(b_j\) is between \(aq_i\) and \(q_i\), then player \(i\) loses if he bids \(aq_i\) and wins if he bids \(q_i\), and in this case he would like to win, because his value for the bundle is at least \(q_i\). (The cases where \(b_j = aq_i\) and \(b_j = q_i\) are handled similarly.)

(d) Suppose that player 2 plays \(s_2(q_2) = \alpha_2 q_2\) for some \(\alpha_2 \geq 1\). Suppose that a player 1 of type \(q_1\) bids \(b_1 \in [0, \alpha_2]\). (There is no need to consider higher bids, because a bid of \(b_1 = \alpha_2\) already wins with probability 1.) Then ignoring ties, which happen with probability zero,
(1) Player 1 wins if $b_1 \geq \alpha_2 q_2$, or equivalently, when $q_2 \leq \frac{b_1}{\alpha_2}$ (so chance $\frac{b_1}{\alpha_2}$).

(2) Conditional on $b_1$ being the winning bid, player 2’s bid is uniformly distributed on $[0, b_1]$, and so her type is uniformly distributed on $[0, \frac{b_1}{\alpha_2}]$. So conditional on $b_1$ being the winning bid, player 1’s expected payment is $\frac{b_1}{2}$ and player 2’s expected type is $\frac{b_1}{2} \alpha_2$.

It follows from (1) and (2) that player 1’s expected utility when he is of type $q_1$ and bids $b_1$ is

$$
\frac{b_1}{\alpha_2} \left( \left( q_1 + \delta \frac{b_1}{2\alpha_2} \right) - \frac{b_1}{2} \right) = \frac{1}{2(\alpha_2)^2} \left( 2\alpha_2 q_1 b_1 - (\alpha_2 - \delta)(b_1)^2 \right).
$$

Maximizing this function over $b_1 \leq \alpha_2$ yields

$$
b_1 = \frac{\alpha_2}{\alpha_2 - \delta} t_1
$$

if this quantity is not bigger than $\alpha_2$, and $\alpha_2$ otherwise. Thus if there is to be a linear equilibrium, it must be that $\frac{\alpha_2}{\alpha_2 - \delta} \leq \alpha_2$, or equivalently, that $\alpha_2 \geq 1 + \delta$, and the corresponding inequality must hold for player 1 as well. We will see that this is indeed the case in the solution we compute.

Equation (*) implies that if a linear equilibrium exists, it must be that $\alpha_1 = \frac{\alpha_2}{\alpha_2 - \delta}$. By symmetry we must also have $\alpha_2 = \frac{\alpha_1}{\alpha_1 - \delta}$. Solving these equations yields $\alpha_1 = \alpha_2 = 1 + \delta$.

Since these values satisfy the inequalities from the last paragraph, they define the unique Nash equilibrium in linear bidding strategies.

(e) A bidder subject to the winner’s curse does not condition on its bid having won when considering its opponent’s type. (But it makes the most sense to assume that it does condition correctly when considering its opponent’s bid.) Thus regardless of its bid, the bidder’s expectation of the opponent’s signal is always $\frac{1}{2}$. Thus if bidder 1 is subject to the winner’s curse, it maximizes

$$
\frac{b_1}{\alpha_2} \left( \left( q_1 + \delta \frac{b_1}{2} \right) - \frac{b_1}{2} \right).
$$

This is accomplished by choosing $b_1 = q_1 + \delta$ regardless of the opponent’s bidding function. In the equilibrium computed in part (iv), bidder 1 instead bids $s_1(q_1) = (1 + \delta)q_1$. Thus, compared to a bidder using this function, a bidder 1 suffering from the winner’s curse bids too high when $q_1 < \frac{1}{2}$. If such a bidder 1 lowered its bid slightly, it would cause it to lose when bidder 2’s bid is $(1 + \delta)q_1$, implying that bidder 2’s type is $q_1 < \frac{1}{2}$. But bidder 1 found it optimal to bid $(1 + \delta)q_1$ under the assumption that the quality of parcel 2 is $\frac{1}{2}$ for sure. Since lowering its bid only causes bidder 1 to lose in certain cases in which parcel 2’s quality is less than $\frac{1}{2}$, bidder 1 must strictly prefer to lower its bid.

By similar logic, a bidder 1 suffering from the “winner’s curse” bids too low when $q_1 > \frac{1}{2}$. In this case, by raising its bid slightly from $(1 + \delta)q_1$, bidder 1 wins the auction in some new cases in which the quality of parcel 2 is greater than $\frac{1}{2}$. Since a bid of $(1 + \delta)q_1$ was optimal under the assumption that the quality of parcel 2 is $\frac{1}{2}$ for sure, bidder 1 must be better off raising its bid. Thus the “winner’s curse” also implies a loser’s curse!
Part III

Wool \( w \) and mutton \( m \) are produced by a competitive market from sheep \( \sigma \) according to the production function \( m = w = \sigma \). In other words, one sheep provides both meat and wool. The market demand for meat and wool is \( D(p_m) = a - bP_m \) and \( D(p_w) = A - BP_w \). Assume that sheep are supplied competitively, and the supply price of sheep is \( C(\sigma) = \sigma \). Assume further that all the increasing costs are external (overgrazing public grass), so that each sheep herder (shepherd) faces constant internal costs.

(a) Find the market price for sheep, wool, and mutton, and market quantity. Illustrate this outcome in a diagram.

(b) Suppose that Mutton International monopolizes the mutton market. Derive the new market price for sheep, wool, and mutton, and market quantity. Illustrate this outcome in a diagram.

Solution sketch: The big idea is that the price of a sheep is paid for by its wool and its mutton.

(a) The increasing costs of raising sheep being external, the supply price of sheep is the marginal cost, namely, \( P_m + P_w = P_s \), or \( (A - \sigma)/B + (a - \sigma)/b = \sigma \). (Let’s ignore corner solutions and tricky problems that arise when vertically summing linear demand curves.) Equilibrium quantity thus solves \((b + B + B\sigma)(A - \sigma)/B = Ab + aB\), and thus is \( \sigma^* = (Ab + aB)/(bB + b + B) \). The market price for sheep, wool, and mutton are accordingly defined from this quantity:

\[
P_m = \frac{(A - (Ab + aB)/(bB + b + B))}{B}
\]

\[
\Rightarrow P_m = \frac{Ab + A - a}{bB + b + B}
\]

(b) Now Mutton International takes the derived supply of meat as a given. Namely, the supply price \( \sigma \) of sheep minus the demand price for wool \((A - \sigma)/B\), or \( \hat{P}_m(\sigma) = \sigma + (\sigma - A)/B \) if \( \sigma > A \); this yields the marginal factor cost \( MFC_m(\sigma) = 2\sigma + (2\sigma - A)/B \). MI solves \( P = MFC_m(\sigma) \), to deduce quantity \( \sigma \) and then computes price by the derived supply curve.
Part IV

Question 1: The Federal Reserve Board in D.C. hires an intern who can choose effort $e \in \{e_L, e_H\}$ at cost 0 and $c$, respectively. Output of the intern’s effort can be low or high $\{q_L, q_H\}$. If the intern exerts effort $e_L$ or $e_H$, then, respectively, the high output is realized with probability $p_L$ or $p_H$. The intern’s utility is $w - c(e)$, where $w$ is the wage and $c(e)$ the cost of the effort. The intern must be paid a non-negative wage. There is no individual rationality constraint. The Fed’s payoff is $q - w$, where $q$ is the output.

(a) Characterize the optimal contract (wages and effort) when effort is not observable.

Suppose now that there are two types of interns, $i \in \{1, 2\}$. The Fed does not observe an intern’s type but knows that the probability of either type is 0.5. For intern $i \in \{1, 2\}$ the cost of exerting effort $e_L$ is 0 and the cost of effort $e_H$ is $c_i$, where $c_2 > c_1$. Interns are identical otherwise. Consider the four profiles of efforts specified below. For each, characterize the wages that a profit maximizing Fed would choose to implement these actions and compute the profits, or show why the actions cannot be implemented. If there are any rents for either type, explain which type enjoys the rents and why.

(b) $e_1 = e_L$ and $e_2 = e_L$?

(c) $e_1 = e_H$ and $e_2 = e_H$?

(d) $e_1 = e_L$ and $e_2 = e_H$?

(e) $e_1 = e_H$ and $e_2 = e_L$?

Solution sketch:

(a) The low effort can be implemented by choosing $w_L = 0$ and $w_H = 0$. Profit is $q_L + p_H(q_H - q_L)$. To implement the high action, choose $w_L = 0$ and $w_H = -\frac{c}{p_H - p_L}$. The Fed’s profit is given by $q_L + p_H\left(q_H - q_L - \frac{c}{p_H - p_L}\right)$.

(b) Choose $w_1^L = 0$ and $w_1^H = 0$ and $w_2^L = 0$ and $w_2^H = 0$. The Fed’s profit is given by $q_L + p_L(q_H - q_L)$.

(c) The principal should set $w_1^L = 0$ and $w_1^H = -\frac{c^2}{p_H - p_L}$ and $w_2^L = 0$ and $w_2^H = -\frac{c^2}{p_H - p_L}$. Both sets of the incentive constraints are satisfied. That is, neither type has an incentive to mimic the other. Second, each type has an incentive to choose the effort intended. Profits for the Fed are $q_L + p_H\left(q_H - q_L - \frac{c^2}{p_H - p_L}\right)$.

(d) Suppose that the intern with high cost is willing to exert high effort, then so is the intern with low cost. Hence, profile $e_1 = e_L$ and $e_2 = e_H$ of effort cannot be implemented.
(e) The incentive constraints for type 1 are

\[ w^1_L + p_H (w^1_H - w^1_L) - c^1 \geq w^1_L + p_L (w^1_H - w^1_L) \geq w^2_L + p_L (w^2_H - w^2_L) \geq w^1_L + p_H (w^2_H - w^2_L) - c^1. \]

The incentive constraints for type 2 are

\[ w^2_L + p_L (w^2_H - w^2_L) \geq w^2_L + p_H (w^2_H - w^2_L) - c^2 \geq w^1_L + p_L (w^1_H - w^1_L) \geq w^1_L + p_H (w^1_H - w^1_L) - c^2. \]

Since the Fed wants type 2 to exert low effort, they can set a constant wage schedule \( w^2_L = w^2_H = w^2 \); and since the Fed wants type 1 to exert high effort they can set \( w^1_L = 0 \). The incentive constraints become

\[ p_H w^1_H - c^1 \geq p_L w^1_H \geq w^2 \geq p_L w^1_H \geq p_H w^1_H - c^2. \]

To satisfy these constraints, set \( w^1_H = \frac{c^1}{p_H - p_L} \) and \( w^2 = \frac{p_L c^1}{p_H - p_L} \). The Fed’s profit is given by

\[ q_L + \frac{p_H + p_L}{2} \left( q_H - q_L - \frac{c^1}{p_H - p_L} \right) \]

The rent for type 2 makes incentivizing type 1 interns to choose H more expensive.