University of Wisconsin
Microeconomics Prelim Exam with Solution Sketches
Monday, June 15, 2015: 9AM - 2PM

- There are four parts to the exam. All four parts have equal weight.
- Answer all questions. No questions are optional.
- Hand in 12 pages, written on only one side.
- Write your answers for different parts on different pages. So do not write your answers for questions in different parts on the same page.
- Please place a completed label on the top right corner of each page you hand in. On it, write your assigned number, and the part of the exam you are answering (I,II,III,IV). Do not write your name anywhere on your answer sheets!
- Show your work, briefly justifying your claims. Some solutions might be faster done by drawing a suitable diagram.
- You cannot use notes, books, calculators, electronic devices, or consultation with anyone else except the proctor.
- Please return any unused portions of yellow tablets and question sheets.
- There are five pages on this exam, including this one. Make sure you have all of them.
- Best wishes!
Part I

1. How would the expected profits of a competitive firm be affected if its input costs were randomly oscillating about a mean, but if the firm were unable to react fast enough to change its production quantity, but adjust its inputs? How would the expected profits of the monopolist with linear demand be affected if the vertical intercept of the demand curve were randomly oscillating about a mean?

Solution: For the first claim, note that inputs are chosen after seeing the realization of factor prices, while the ultimate quantity is not optimized. So the profit function is not relevant here. The question is explicitly on the fixed quantity optimization. But since the cost function is a concave function of input prices, for fixed $q$, expected costs fall; therefore, volatility raises expected profits.

Next, consider the monopoly problem. One possible interpretation of the question is that $Q$ must be chosen before seeing the realization of $A$. This trivializes the question: One acts as if $A$ is its expected value, and behavior is invariant. What was intended was the more interesting question in which $Q$ can be chosen after seeing the realization of $A$.

First, proceed by brute force: With linear costs $C(Q) = B + bQ$ — or zero, $B = b = 0$ — profits $\Pi(A|Q) = Q(A - Q) - B - bQ = (A - b)Q - Q^2 - B$ are maximized at $Q = (A - b)/2$. So the maximum profits $\Pi(A) = (A - b)^2/A - B$ are strictly convex in $A$, and so mean zero noise raises expected profits, by Jensen’s inequality.

But this generally holds for any costs (this harder proof is bonus): For since profits are linear in $A$ for each $Q$, the upper envelope of profits $\Pi(A|Q)$ as we optimize across $Q$ is convex in $A$ — just as in the standard proof of the convexity in prices of the profit function for a competitive firm. (Another approach is to use the Envelope Theorem; see if you can figure it out.) So volatility raises expected profits, by Jensen’s inequality.

2. Tono owns $A > 100$ continuously divisible shares of a firm. If he is left with a quantity $a \leq A$ of shares, his salvage value is $V(a)$, where $V(0) = 0$, $V'(0^+) > 0$, with $V' > 0 > V''$ on $(0, \infty)$.

(a) Suppose that $A$ is actually random, equal to $\bar{A} + \epsilon$, where $\epsilon$ is random with $E[\epsilon] = 0$ and $\sigma^2 = 0.1$. What approximate nonrandom level of shares $\bar{A}$ would leave Tono just as happy as with $A$.

Solution: Using the second order Tayor series expansion of Pratt (1964), we easily have $\bar{A} \approx \bar{A} + \frac{1}{2} \sigma^2 V''(\bar{A})/V'(\bar{A}) = \bar{A} + \frac{1}{20} V''(\bar{A})/V'(\bar{A})$.

(b) Assume now that $A$ is certain. Assume that buyer offer to pay $p$ per share for any amount of the firm. Derive a formula for Tono’s supply of shares $Y(p, A)$ to the buyer as a function of $(p, A)$. Be rigorous.

Solution: Tono solves $\max_y [py + V(A - y)]$ s.t. $0 \leq y \leq A$. As $V$ is strictly concave and the constraints are linear, the FOC is necessary and sufficient for a maximum. Since at most one constraint binds, the Kuhn-Tucker constraint qualification is met. If the multipliers on $0 \leq y$ and $y \leq A$ are respectively $\lambda_1, \lambda_2 \geq 0$, then the FOC is $p - V'(A - y) + \lambda_1 - \lambda_2 = 0$. By complementary
slackness, if \( y = A \), then \( p - V'(0+) \geq 0 \), and if \( y = 0 \), then \( \lambda_1 > 0 \), and if \( y = 0 \) then \( \lambda_2 > 0 \) and thus \( p - V'(A) \leq 0 \). Otherwise, the multipliers vanish, and thus \( p = V'(A - y) \). For any share price \( p \), the optimal supply is thus:

\[
Y(p, A) = \begin{cases} 
\max(A - (V')^{-1}(p), 0) & \text{for } p \leq V'(0+) \\
A & \text{for } p > V'(0+) 
\end{cases}
\]

(c) Assume also that \( V''' > 0 \) on \((0, \infty)\). When positive, show that the supply \( Y(p, a) \) is increasing and strictly concave in \( p \). Plot supply as a function of \( A \) and as a function of \( p \).

Solution: Note that supply is \( Y(p, A) = A - (V')^{-1}(p) \) when positive. Since \( V' \) is decreasing and convex, its inverse is increasing and convex in \( p \). So

- As a function of \( p \), supply is zero when \( p < V'(A) \), increasing and concave in \( p \) when \( p \in [V'(A), V'(0+)] \), and leveling off at \( A \) when \( p > V'(0) \).

- As a function of \( A \), supply is 0 for high \( A \leq (V')^{-1}(p) \) (so that \( V'(A) \geq p \), but rising dollar for dollar in \( A \) for larger \( A > (V')^{-1}(p) \) (and so \( V'(A) < p \)); it equals \( A \) when \( p > V'(0+) \)
Part II

Two players choose actions from the unit interval. Player $i$’s von-Neumann Morgenstern utility, expressed as a function of his action $x_i \in [0, 1]$ and his opponent’s action $x_j \in [0, 1]$, is

$$u_i(x_i, x_j) = \begin{cases} 
(\theta_i + 3x_j - 4x_i)x_i & \text{if } x_j < \frac{2}{3}, \\
(3x_j - 2)x_i & \text{if } x_j \geq \frac{2}{3}.
\end{cases}$$

First, consider the normal form game $G$, in which $\theta_1 = \theta_2 = 2$.

1. Find all pure Nash equilibria of $G$.

Next, consider the Bayesian game $BG$ in which player 1 is of type $\theta_1 = 1$ or $\theta_1 = 3$, each with probability $\frac{1}{2}$, and player 2 is of type $\theta_2 = 2$ with probability 1.

2. Find all pure Nash equilibria of $BG$. (Hint: Define a map whose fixed points are the equilibrium strategies of player 2.)

Finally, consider the Bayesian game $BG'$ in which player 1 is of type $\theta_1 = -1$ or $\theta_1 = 5$, each with probability $\frac{1}{2}$, and in which player 2 is of type $\theta_2 = 2$ with probability 1.

3. Find all pure Nash equilibria of $BG'$.

1. If $\theta_i \in [0, 6]$, player $i$’s pure best response correspondence is

$$b_i(x_j) = \begin{cases} 
\{ \frac{\theta_i + 3x_j}{8} \} & \text{if } x_j < \frac{2}{3}, \\
[0, 1] & \text{if } x_j = \frac{2}{3}, \\
\{1\} & \text{if } x_j > \frac{2}{3}.
\end{cases} \quad (1)$$

In the present case, $\theta_1 = \theta_2 = 2$.

Clearly the Nash equilibria with $x_2 \geq \frac{2}{3}$ are $(\frac{2}{3}, \frac{2}{3})$ and $(1, 1)$. To check for the remaining equilibria, we look for $x_2 \in [0, \frac{2}{3})$ that satisfy

$$2 + 3 \left( \frac{2+3x_2}{8} \right) = x_2.$$ 

The unique solution is $x_2 = \frac{2}{5}$, which is in $[0, \frac{2}{3})$. Since $b_1(\frac{2}{5}) = \frac{2}{5}, \left( \frac{2}{5}, \frac{2}{5} \right)$ is a Nash equilibrium.

2. Write $b_{1\ell}$ and $b_{1h}$ for the best response correspondences of $\theta_1 = 1$ and $\theta_1 = 3$. Both are given by (1) with appropriate choices of $\theta_i$. A key point to be used below is that if $x_2 < \frac{2}{3}$, then $b_{1\ell}(x_2) < \frac{2}{3}$ and $b_{1h}(x_2) < \frac{2}{3}$. 

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Write \( x_{1f} \) and \( x_{1h} \) for the actions of the two types of player 1. Then player 2’s expected payoff function in \( BG \) is

\[ U_2(x_{1f}, x_{1h}) = \begin{cases} \frac{1}{2} (2 + 3x_{1f} - 4x_2)x_2 + \frac{1}{2} (2 + 3x_{1h} - 4x_2)x_2 & \text{if } x_{1f} < \frac{2}{3} \text{ and } x_{1h} < \frac{2}{3}, \\ \frac{1}{2} (2 + 3x_{1f} - 4x_2)x_2 + \frac{1}{2} (3x_1 - 2)x_2 & \text{if } x_{1f} < \frac{2}{3} \text{ and } x_{1h} ≥ \frac{2}{3}, \\ \frac{1}{2} (3x_{1f} - 2)x_2 + \frac{1}{2} (2 + 3x_{1h} - 4x_2)x_2 & \text{if } x_{1f} ≥ \frac{2}{3} \text{ and } x_{1h} < \frac{2}{3}, \\ \frac{1}{2} (3x_{1f} - 2)x_2 + \frac{1}{2} (3x_1 - 2)x_2 & \text{if } x_{1f} ≥ \frac{2}{3} \text{ and } x_{1h} ≥ \frac{2}{3}, \end{cases} \]

where \( x_1 = \frac{1}{4}(x_{1f} + x_{1h}) \).

Let \( B_2: [0, 1] \times [0, 1] \rightarrow [0, 1] \) be player 2’s best response correspondence in the Bayesian game. Then

\[ B_2(x_{1f}, x_{1h}) = \begin{cases} \{ \frac{2+3x_1}{8} \} & \text{if } x_{1f} < \frac{2}{3} \text{ and } x_{1h} < \frac{2}{3}, \\ [0, 1] & \text{if } x_{1f} = \frac{2}{3} \text{ and } x_{1h} = \frac{2}{3}, \\ \{ 1 \} & \text{if } x_{1f} ≥ \frac{2}{3} \text{ and } x_{1h} ≥ \frac{2}{3}, \text{ not both with equality}, \\ \{ \frac{3x_1}{4} \} & \text{otherwise.} \end{cases} \ NUMBER \tag{2} \]

The Nash equilibrium strategies of player 2 are the fixed points of the set-valued map \( β_2: [0, 1] \rightarrow [0, 1] \) defined by

\[ β_2(x_2) = B_2(b_{1f}(x_2), b_{1h}(x_2)). \]

If \( x_2 < \frac{2}{3} \), then (dropping the set brackets) \( b_{1f}(x_2) = \frac{1+3x_j}{8} < \frac{2}{3} \) and \( b_{1h}(x_2) = \frac{3+3x_j}{8} < \frac{2}{3} \). Therefore \( x_1 = \frac{2+3x_2}{8} \). Thus the first case of (2) and the same calculation as in part (ii) shows that \( x_2 = \frac{2}{3} \) is a fixed point of \( β_2 \). This corresponds to the equilibrium \((x_{1f}, x_{1h}, x_2) = (\frac{11}{40}, 1, \frac{2}{3})\).

If \( x_2 > \frac{2}{3} \), then \( b_{1f}(x_2) = b_{1h}(x_2) = 1 \), so the only corresponding fixed point is \( x_2 = 1 \), corresponding to the equilibrium \((1, 1, 1)\).

If \( x_2 = \frac{2}{3} \), then \( b_{1f}(x_2) = b_{1h}(x_2) = [0, 1] \). For \( x_2 = \frac{2}{3} \) to be a best response for player 2, the actions of player 1 must be in the second or fourth case of (2). The second case corresponds to the equilibrium \((\frac{2}{3}, \frac{2}{3}, \frac{2}{3})\). For the best response in the fourth case to equal \( \frac{2}{3} \), we need \( x_1 = \frac{8}{9} \), since this is inconsistent with one type of player 1 choosing an action below \( \frac{2}{3} \); this case does not lead to any equilibria.

3. Write \( b_{1f} \) and \( b_{1h} \) for the best response correspondences of the types \( θ_1 = -1 \) and \( θ_1 = 5 \). The latter is again given by (1), but the former requires an additional case:

\[ b_{1f}(x_2) = \begin{cases} \{ 0 \} & \text{if } x_j ≤ \frac{1}{3}, \\ \{ -\frac{1+3x_j}{8} \} & \text{if } x_j ∈ (\frac{1}{3}, \frac{2}{3}), \\ [0, 1] & \text{if } x_j = \frac{2}{3}, \\ \{ 1 \} & \text{if } x_j > \frac{2}{3}. \end{cases} \]
Note that \( b_{1h}(\frac{1}{3}) = \frac{2}{3} \) and that \( b_{1h}(x_2) > \frac{2}{3} \) whenever \( x_2 \in (\frac{1}{3}, \frac{2}{3}) \).

Player 2’s best response correspondence is again given by (2), and we again look for fixed points of (3). As before, the equilibria with \( x_2 \geq \frac{2}{3} \) are \(((x_{1\ell}, x_{1h}), x_2) = ((1, 1), 1)\) and \(((\frac{2}{3}, \frac{2}{3}), \frac{2}{3})\).

We now consider equilibria with \( x_2 < \frac{2}{3} \), which could come from the first, second, or fourth cases of (2). In the first case, \( x_2 \) must be in \([\frac{1}{3}, \frac{1}{2}]\); thus \( b_{1h}(x_2) > \frac{2}{3} \), which contradicts being in the first case. For the second case, note that \( \frac{2}{3} \in b_{1\ell}(x_2) \) only if \( x_2 = \frac{2}{3} \), contradicting the initial assumption.

Thus we consider the fourth case. Since \( x_2 < \frac{2}{3} \), \( b_{1\ell}(x_2) < \frac{2}{3} \), so to be in this case we must have \( b_{1h}(x_2) \geq \frac{2}{3} \), and thus \( x_2 \geq \frac{1}{3} \).

We divide into two subcases. If \( x_2 \in (\frac{1}{3}, \frac{2}{3}) \), then we are in the second case of (4), so \( b_{1\ell}(x_2) = \frac{-1+3x_j}{8} \) and \( b_{1h}(x_2) = \frac{5+3x_j}{8} \), and hence \( \bar{x}_1 = \frac{2+3x_j}{8} \). Thus a fixed point of (3) in this range must satisfy

\[
3 \left( \frac{2+3x_j}{8} \right) = x_2.
\]

The unique solution is \( x_2 = \frac{6}{23} < \frac{1}{3} \). Thus there is no equilibrium with \( x_2 \in (\frac{1}{3}, \frac{2}{3}) \).

If instead \( x_2 \in [\frac{1}{3}, \frac{1}{2}] \), we are in the first case of (4), so \( b_{1\ell}(x_2) = 0 \) and \( b_{1h}(x_2) = \frac{5+3x_j}{8} \), and hence \( \bar{x}_1 = \frac{3+3x_j}{16} \). Thus a fixed point of (3) in this range must satisfy

\[
3 \left( \frac{5+3x_j}{16} \right) = x_2.
\]

The unique solution is \( x_2 = \frac{3}{11} \in [\frac{1}{9}, \frac{1}{3}] \), yielding the equilibrium \(((x_{1\ell}, x_{1h}), x_2) = ((0, \frac{8}{11}), \frac{3}{11})\).
Part III

1. Assume linear demand and that all firms have zero costs. Rank these cases according to total profits and market quantity: (i) a monopoly; (ii) a Stackelberg monopoly (one first mover, one second mover); (iii) a Cournot duopoly; (iv) a monopoly with a competitive fringe (a first mover, and free entry of a continuum of second movers).

Solution: The answer is the same for any linear demand, so assume \( P(Q) = A - Q \). Profits are price times quantity, or \( Q(A - Q) \). So it suffices to rank profits. Clearly, a monopolist maximizes profits, and so chooses \( Q = A/2 \). So (i) is the least quantity and maximum profits. The competitive fringe earns zero profits and so \( P = 0 \). So (iv) is the maximum quantity and minimum profits. A Stackelberg leader could choose to do what a Cournot duopolist does, and if he did, the follower would choose the Cournot quantity too. So by revealed preference, the profits of the Stackelberg leader are higher than he gets in a Cournot duopoly. If joint profits were higher, then the Cournot could flip a coin and pick either to be the Stackelberg mover. So joint profits in (iii) are higher, and thus quantity lower, than in (ii). But this violates the quasiconcavity of profits in quantity in each Cournot problem. In sum, the profits are ranked (i) > (iii) > (ii) > (iv), and quantities inversely so.

2. Assume linear demand. Let \( K \) identical firms with zero costs be in Cournot competition. Assume that \( J \) firms can merge, and that after the merger, they act as a single firm (or a cartel) in Cournot competition with the \( K - J \) other firms. For what values of \( J \) and \( K \geq J \) for which the merged firms as a cartel earn higher total profits than they did before the merger?

Solution: Again, WLOG assume \( P(Q) = A - Q \). Assume total quantity \( Q \) from all other firms. Cournot’s assumption yields profits \( \max_q (A - q - Q)q \). The FOC is thus \( A - 2q - Q = 0 \) for all firms. Thus, we have \( Q = (K - 1)q = A - 2q \), or \( q = A/(K+1) \). Thus, as one of \( K \) equal firms, each firm earns profits

\[
[A/(K+1)][A - KA/(K+1)] = A^2/(K+1)^2
\]

After a merger by \( J \) firms that act as a cartel, the merged firms’ profits total profits, as one of \( K - J + 1 \) equal firms, are \( 1/J[A/(K-J+2)][A - (K-J+1)A/(K-J+2)] \). So the question is when is:

\[
[1/(K - J + 2)][1 - (K - J + 1)/(K - J + 2)] > J/(K+1)^2
\]

\[
J(K - J + 2)^2 < (K + 1)^2 \iff \sqrt{J}(K + 1 - J + 1) < K + 1
\]

This is equivalent to \( K^2 - 2(J - 1)K + (J^2 - 3J + 1) > 0 \). By the quadratic formula, with the discriminant \( 4(J - 1)^2 - 4(J^2 - 3J + 1) = 4J + 4 - 4 = 4J \), yields \( K > J_1 + \sqrt{J} \).

PS Let’s do an acid test to check for errors. We know that if all \( J = K \) firms merge, their profits rise. So plugging \( J = K \) must satisfy the inequality. It does.
3. Consider three binary action games: (i) the prisoner’s dilemma; (ii) the game of chicken; (iii) the battle of the sexes; and (iv) matching pennies. In which of games (i)–(iv) would a player strictly prefer the role of spy, namely, where it is commonly understood that the spy can act after seeing the pure action chosen by the non-spy?

Solution: The key idea is that spying formally converts the game into an extensive form game of perfect information in which the non-spy moves first. So the question asks which games have a strict advantage for the second mover. The answer is only matching pennies. The first mover secures his preferred outcome in games (ii) and (iii), and neither first nor second mover secure an advantage in game (i).
Part IV

Consider the following model of job market signalling. The players are a worker and two firms. The worker’s type $\theta \in \Theta = \{\theta_L, \theta_H\}$, $0 < \theta_L < \theta_H$, is his ability level. He is type $\theta_H$ with probability $p_H$.

The game proceeds as follows: [0] The worker learns his type; [1] The worker chooses an education level $e \in [0, \infty)$; [2] The two firms observe the worker’s education level and simultaneously make wage offers $w_1, w_2 \in [\theta_L, \theta_H]$; [3] The worker then chooses which offer to accept, if any.

Firm $i$’s payoff for hiring a worker of type $\theta_A \in \Theta$ at wage $w$ is $u_i(w, \theta_A) = \theta_A - w$. Its payoff for not hiring a worker is 0.

The payoff of a type $\theta_A \in \Theta$ worker is $u_A(w, e) = w - c_A(e)$, where $c_A(\cdot)$ is differentiable, increasing, and strictly convex with $c_A(0) = 0$. We also assume that $c'_L(e) > c'_H(e)$ for $e > 0$.

1. Describe each player’s pure strategy set in this game.

2. State the requirements that define a pure perfect Bayesian equilibrium with common beliefs in this game.

3. Derive the set worker strategies that can arise in a separating equilibrium. (Diagrams may be helpful here.) For each worker strategy you identify, fully describe an equilibrium in which the worker chooses this strategy.

4. Define a weak notion of forward induction that allows you to eliminate all but one pooling equilibrium, and prove that it does so.

5. Derive the set worker strategies that can arise in a pooling equilibrium. (Diagrams may also be helpful here.) For each worker strategy you identify, fully describe an equilibrium in which the worker chooses this strategy.

6. Define a notion of forward induction that rules out all pooling equilibria, and prove that it does so.

7. Draw a diagram representing a pooling equilibrium that could also be ruled out by the criterion you used in part (iv), and explain why the criterion rules out the equilibrium.

8. Draw a diagram representing a pooling equilibrium that cannot be ruled out by the criterion you used in part (iv), and explain why the criterion does not rule out the equilibrium. (Hint: Assume that $p_H$ is close to 1.)

Solutions:

1. The worker’s pure strategy set is \{functions from $\Theta$ to $[0, \infty)$\} $\times$ \{functions from $\Theta \times [0, \infty) \times [\theta_L, \theta_H]^2$ to \{accept, reject\} \}. Each firm’s pure strategy set is a function from $[0, \infty)$ to $[\theta_L, \theta_H]$. 

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2. The common beliefs assumption says that there is a function \( \mu : \mathbb{R}_+ \to \Delta \Theta \) describing both firms’ beliefs about the worker’s type. In a pure perfect Bayesian equilibrium with common beliefs, \((e_L, e_H, w_1(\cdot), w_2(\cdot), \mu(\cdot))\):

(I) In the last stage, the worker chooses optimally among wage offers (i.e., works for a firm whose wage offer is highest);

(II) In the second-to-last stage, having observed the worker’s education choice \( e \), the firms choose wages \( w_1(e) \) and \( w_2(e) \) optimally given their common belief \( \mu(e) \). (In view of the competition between the firms, both will choose wages equal to the worker’s expected ability.)

(III) Each worker type chooses his education level optimally given (I) and (II).

(IV) Beliefs \( \mu(e) \) are given by conditional probabilities when possible.

3. In a separating equilibrium, the two types choose different education levels, \( e_L \neq e_H \). Thus \( \mu_L(e_L) = 1 \) and \( \mu_H(e_H) = 1 \), implying that \( w_1(e_L) = w_2(e_L) = \theta_L \) and \( w_1(e_H) = w_2(e_H) = \theta_H \). Therefore \( e_L = 0 \) in equilibrium, since a low ability worker who puts in a positive effort will still be paid \( \theta_L \). For each type to prefer its own effort level, we must have

\[
\theta_L - c_L(0) \geq \theta_L - c_L(e_H) \quad \iff \quad c_L(e_H) \geq \theta_H - \theta_L \\
\theta_H - c_H(e_H) \geq \theta_L - c_H(0) \quad \iff \quad c_H(e_H) \leq \theta_H - \theta_L
\]

That is, equilibrium requires that

\[c_L(e_H) \geq \theta_H - \theta_L \geq c_H(e_H).\] (5)

Single crossing and the fact that \( c_L(0) = c_H(0) \) implies that \( c_L(e) > c_H(e) \) for all \( e > 0 \). Thus if we define \( \underline{e} \) and \( \overline{e} \) by \( c_L(\underline{e}) = \theta_H - \theta_L \) and \( c_H(\overline{e}) = \theta_H - \theta_L \), then \( \underline{e} < \overline{e} \), and all \( e_H \in [\underline{e}, \overline{e}] \) satisfy (5), and so are possible in equilibrium.

To complete the description of equilibrium we must specify out-of-equilibrium beliefs. One possibility is to choose \( \mu_L(e) = 1 \) for all \( e \notin \{0, e_H\} \).

4. We can refine the set of equilibria by applying a weak form of forward induction based on removal of conditionally dominated strategies. If firms only choose wages in \([\theta_L, \theta_H] \), then \( \theta_L \) must be better off playing \( e = 0 \) and obtaining a payoff of at least \( \theta_L \) than playing \( e > \underline{e} \) and obtaining a payoff of at most \( \theta_H - c_L(e) < \theta_H - c_L(\underline{e}) = \theta_L \). This leads to the forward induction requirement that \( \mu_L(e) = 0 \) for all \( e > \underline{e} \) that are not dominated for type \( \theta_H \). This rules out every equilibrium with \( e_H > \underline{e} \), since type \( \theta_H \) could deviate to \( e_H' \in (0, e_H) \), pay a lower cost, but still receive wage \( \theta_H \). Thus only the separating equilibria with \( e_H^* = \underline{e} \) survive.

5. In a pooling equilibrium, \( e_L = e_H = e^* \). Thus \( \mu_H(e^*) = p_H \) and \( w_i(e^*) = (1 - p_H)\theta_L + p_H\theta_H = \theta \).
7. The pooling equilibrium in the figure below could be eliminated by the refinement based on elimination of conditionally dominated strategies. For type $\theta_L$, any $e > e'$ is dominated by choosing $e_L = 0$, since $(e, \theta_H)$ lies below $\theta_L$’s indifference curve through $(0, \theta_L)$. Thus the refinement requires $\mu_L(e) = 0$ for all $e > e'$ that are not dominated for type $\theta_H$, leading type $\theta_H$ to deviate as before.

6. We impose forward induction here by way of the Cho-Kreps criterion. Fix an equilibrium with payoffs $u_L^*$ and $u_H^*$. For $T \subseteq \Theta$, define $W_T(e)$ to be the set of equilibrium wages that are possible given education level $e$ if the firms’ beliefs satisfy $\sum_{\theta_A \in T} \mu_A(e) = 1$. Then

$$D(e) = \left\{ \theta_A: u_A^* > \max_{w \in W_A(e)} u_A(w, e) \right\}$$

is the set of types for whom education level $e$ is equilibrium dominated. If for some $e \neq e^*$ with $D(e) \neq \emptyset$, and some type $\theta_B \in \emptyset$, we have

$$u^*_B < \min_{w \in W_B - D(e)} u_B(w, e),$$

then the component fails the Cho-Kreps criterion.

The Cho-Kreps criterion rules out all pooling equilibria. Fix a pooling equilibrium with common effort level $e^* \in [0, \bar{e})$, so that $\bar{\theta} - c_L(e^*) \geq \bar{\theta} - c_L(0)$ and $\bar{\theta} - c_H(e^*) \geq \bar{\theta} - c_H(0)$. Define $e' > e^*$ by $\bar{\theta} - c_L(e^*) = \bar{\theta} - c_L(e')$ and $e'' > e^*$ by $\bar{\theta} - c_H(e^*) = \bar{\theta} - c_H(e'')$. The single crossing property implies that $e'' > e'$. Now if $e \in (e', e'')$, then $u_H^* > \bar{\theta}_H - c_L(e)$ and $u_H^* < \bar{\theta}_H - c_H(e)$. Thus $D(e) = \{ \theta_L \}$ and $\emptyset - D(e) = \{ \theta_H \}$. But if $\mu_H(e) = 1$, then both firms choose $w_i(e) = \theta_H$. Thus since $u_H^* < \bar{\theta}_H - c_H(e)$, type $\theta_H$ prefers to deviate to $e$, breaking the equilibrium.

Specify $\mu_L(e) = 1$ for $e \neq e^*$, so that $w_i(e) = \theta_L$ for such $e$. Then pooling equilibrium requires

$$\bar{\theta} - c_L(e^*) \geq \bar{\theta} - c_L(0) \iff c_L(e^*) \leq \bar{\theta} - \bar{\theta}_L, \quad (6)$$

$$\bar{\theta} - c_H(e^*) \geq \bar{\theta} - c_H(0) \iff c_H(e^*) \leq \bar{\theta} - \bar{\theta}_L. \quad (7)$$

Single crossing and the fact that $c_L(0) = c_H(0)$ implies that $c_L(e) > c_H(e)$ for all $e > 0$, and so (6) implies (7). Let $\bar{e}$ be the education level that makes (6) bind. Then all $e^* \in [0, \bar{e})$ are possible pooling equilibrium education levels.
8. The pooling equilibrium in the figure below cannot be eliminated by the refinement based on elimination of conditionally dominated strategies. Effort levels in $e \in (e', e'')$ are not dominated for type $\theta_L$, since the points $(e, \theta_H)$ all lie above $\theta_L$’s indifference curve through $(0, \theta_L)$. We can only rule out type $\theta_L$ choosing these effort levels if we compare his possible payoffs from choosing them to his equilibrium payoff $(e^*, \bar{\theta})$, as we do under the Cho-Kreps criterion.